A CHARACTERIZATION OF REFLEXIVE BANACH SPACES

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ABSTRACT. A Banach space Z is not reflexive if and only if there exist a closed separable subspace X of Z and a convex closed subset Q of X with empty interior which contains translates of all compact sets in X . If, moreover, Z is separable, then it is possible to put $X = Z$.

We consider the following problem: When does a Banach space contain a closed convex set Q with empty interior which contains a translate of any compact set in X ? The basic example of such a Banach space is the space $C(\mathcal{K})$ of continuous functions on a compact infinite space $\mathcal{K}$. Indeed, it is enough to choose a point $p \in \mathcal{K}$ which is not isolated and define Q as the set of all functions in $C(\mathcal{K})$ which attain their minima at p . Since p is not isolated, Q has empty interior. If K is a compact subset of $C(\mathcal{K})$, then by the Banach-Dieudonné theorem [3] there exists a sequence $\{f_n\}$ of functions in $C(\mathcal{K})$ converging to zero such that K is contained in its closed convex hull. If we define the function g by

$$g(t) := \sup\{|f_n(t) - f_n(p)| : n \in N\} \text{ for } t \in \mathcal{K},$$

then it is easy to check that g is continuous and each function $g + f_n$ is in Q . Consequently, since Q is convex, the translate $g + K$ is contained in Q .

If a Banach space Z can be mapped linearly onto a Banach space X containing the required set Q , then Z also contains such a set. Namely, by the open mapping theorem, it is enough to take the preimage of Q . Therefore, for example, ℓ_1 contains the required set because it can be mapped onto any separable Banach space, in particular, $C[0, 1]$.

In this note we show that, in fact, any separable nonreflexive Banach space X contains a closed convex set with empty interior which contains a translate of any compact set in X .

Borwein and Noll observed in [1] that there exist a convex continuous function on the space c_0 of null sequences and a closed subset Q of c_0 which is not a Haar null set so that f fails to be Fréchet differentiable on Q . They define f as the distance from the positive cone $Q := \{\{x_n\} \in c_0; x_n \geq 0, n = 1, 2, \dots\}$. As Q has no interior points, f fails to be Fréchet differentiable at all points of Q . The set Q contains a translate of any compact set in c_0 , and, therefore, for any

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probability Borel measure μ on c_0 there exists some $x \in c_0$ such that $\mu(Q+x) > 0$. Consequently, Q is not Haar null (for the definition see [2]). They conjecture in [1] that also in ℓ_2 there exists a closed convex set C with empty interior which contains a translate of any compact set. We show that this is not the case in any reflexive Banach space, but on the other hand every nonreflexive Banach space has a closed subspace containing such a set.

By B_X we denote the open unit ball of a Banach space X , and $B_X(x, r)$ is the usual notation for the open ball with center x and diameter r ; the subscript will be often omitted. We denote the closure of a set A by $\bar{A}$ or $\text{cl}A$.

We will make use of the following variation of the Banach-Dieudonné theorem: Let X be a Banach space, K a compact subset of $B_X(0, c)$ and E a dense subset of $B_X(0, 2c)$. Then there exists a sequence $\{F_n\}$ of finite sets in E so that

$$(1) \quad K \subset \text{cl} \left(\sum_{n=1}^{\infty} 2^{-n} F_n \right).$$

This follows from the fact that there exist a sequence $\{F_n\}$ of finite sets in E and a sequence of compact sets $\{K_n\}$ in $\bar{B}_X(0, c)$ so that

$$(2) \quad K \subset \sum_{i=1}^n 2^{-i} F_i + 2^{-n} K_n \quad \text{for } n \in \mathbb{N}.$$

Indeed, if $n = 1$, choose $F_1 \subset E$ so that $2^{-1}F_1$ is a $\frac{c}{2}$ -net for K . Then the set

$$K_1 := 2 \left((K - 2^{-1}F_1) \cap \bar{B}(0, \frac{c}{2}) \right)$$

is a compact subset of $\bar{B}(0, c)$ and $K \subset 2^{-1}F_1 + 2^{-1}K_1$. Now we can continue by induction. Suppose that F_i and K_i for $i = 1, \dots, n$ so that (2) holds have been already constructed. Choosing $F_{n+1} \subset E$ so that $2^{-1}F_{n+1}$ is a $\frac{c}{2}$ -net for the set K_n and defining

$$K_{n+1} := 2 \left((K_n - 2^{-1}F_{n+1}) \cap \bar{B}(0, \frac{c}{2}) \right)$$

completes the proof. The following lemmata are possibly not the most efficient way to our main result, but we think that they may be of independent interest.

Lemma 1. *Let Z be a Banach space, U an open convex subset of Z and f a continuous real valued function defined on U . Then, either f is affine or the convex hull G of the graph of f has nonempty interior.*

Proof. Suppose that f is not affine. Then there exist x and y in U such that $1/2(f(x) + f(y)) \neq f((x+y)/2)$. Define $z_0 := (x+y)/2$ and $c := (f(x) + f(y))/2$. We can suppose by replacing f by $-f$ and adding a constant, if necessary, that

$$(f(x) + f(y))/2 - f(z_0) = \alpha > 0 \quad \text{and} \quad f(z_0) > 0.$$

Choose some $\varepsilon > 0$ so that $0 < f(v) < f(z_0) + \alpha/2$ for every $v \in Z$ for which $\|v - z_0\| < \varepsilon$. Clearly the interior of the cone cap

$$\begin{aligned} M &:= \{x_{z,t} = t(z, 0) + (1-t)(z_0, c) : z \in Z, \|z - z_0\| < \varepsilon, 0 \leq t \leq \alpha/(2c)\} \\ &\subset Z \times \mathbb{R} \end{aligned}$$

is nonempty. Let some $x_{z,t} \in M$ be given, we will show that $x_{z,t} \in G$. Consider the function

$$g(s) := (1-s)c - f((1-s)z_0 + sz), \quad t \leq s \leq 1.$$

The function g is continuous, $g(t) > 0$, and $g(1) < 0$. Therefore, there exists some $r \in (t, 1)$ for which $g(r) = 0$. Hence, $x_{z,r}$ is contained in the graph of f and since

$$x_{z,t} = \frac{t}{r}x_{z,r} + (1 - \frac{t}{r})(z_0, c)$$

we have $x_{z,t} \in G$. $\square$

We say that a convex subset Q of a Banach space X is spanning if it contains a line segment in every direction, that is $X = \bigcup_{t>0} t(Q - Q)$. Observe that if a convex set Q contains a translate of every finite subset of the unit ball, then Q is spanning. If Q contains translates of all compact sets in X (or, for that matter, of all line segments), then $X = Q - Q$. Indeed, if $x \in X$ is given, then there exists $z \in X$ so that $[z, z+x] \subset Q$, and $x = z+x-z \in Q - Q$.

Lemma 2. *Suppose that X is a Banach space and $Q \subseteq X$ is a bounded, closed and convex set with empty interior that is also spanning. Then for any compact subset K of X it follows that $Q + K$ also has empty interior.*

Proof. First, we show that $Q \cap H$ is nowhere dense in H if H is any closed hyperplane. Suppose that $x^* \neq 0$, $w \in H = \{x^* = a\}$, $\delta > 0$ and

$$B(w, \delta) \cap H \subseteq Q \cap H.$$

Choose some $y \in X$ such that $\langle x^*, y \rangle > 0$. Since Q is spanning there exist $t > 0$, u and v , both in Q , so that $t(u-v) = y$. It follows that $\langle x^*, u-v \rangle > 0$ and one of u or v , say u , is not in H . It is routine to check that the convex hull of $\{u\} \cup (B(w, \delta) \cap H)$ has an interior point relative to X (try $\frac{1}{2}(u+w)$) which contradicts the fact that Q has no interior. Suppose that $H \subseteq X$ is a closed hyperplane, $u \in X$, $x \notin H$ and suppose that $h^* \in H^*$. Then the set $\{y + \langle h^*, y \rangle x + u : y \in H\}$ is a hyperplane in X and the transformation $y \mapsto y + \langle h^*, y \rangle x + u$ is an affine homeomorphism. If $x \in X$, then the set $Q' = Q + [-x, x]$ is also bounded, closed, convex and spanning. We will show that it has empty interior. Suppose that the interior of Q' is nonempty. Then $x \neq 0$; choose $x^* \in X^*$ so that $\langle x^*, x \rangle > 0$. Let P be the projection on X whose image is the kernel H of x^* and whose kernel is the span of x . The open mapping theorem says that $P(Q) = P(Q')$ has nonempty interior in H . Suppose that $w \in H$, $\delta > 0$ and $B(w, \delta) \cap H \subseteq P(Q)$. For $z \in B(w, \delta) \cap H$ define

$$f(z) := \inf\{t : z + tx \in Q\}.$$

It is easy to see that f is bounded and convex, hence continuous. The mapping $(z, t) \mapsto z + tx$ is an isomorphism from $H \times \mathbb{R}$ onto X which maps the graph of f onto the set $\{z + f(t)x : z \in B(w, \delta) \cap H\} \subset Q$. Because Q has empty interior, Lemma 1 implies that f must be affine, and we shall show that this leads to a contradiction. Since it is defined on an open convex subset of H , there exists an $h^* \in H^*$ and a real number b such that $f(z) = \langle h^*, z \rangle + b$. Finally,

$$\{z + \langle h^*, z \rangle x + bx : z \in B(w, \delta) \cap H\} \subseteq Q$$

and this means that Q contains a relatively open subset of a hyperplane, which is a contradiction. By induction, given $x_1, x_2, \dots, x_n \in X$ we have that

$$Q + [-x_1, x_1] + \dots + [-x_n, x_n]$$

has no interior point. The case of an arbitrary compact set K can be verified by an application of (1). We give a few details. Suppose the interior of $Q + K$ is nonempty. By translating $Q + K$ if necessary we can suppose that $B(0, r) \subset Q + K$

for some $r > 0$. Choose a sequence $\{F_n\}$ of finite subsets of a ball in X so that (1) holds. Choose $n_0 \in \mathbb{N}$ so that

$$(3) \quad \sum_{i=n_0}^{\infty} 2^{-i} F_i \subset B(0, r/4).$$

Because the interior of the closed and convex set $Q_0 := Q + \sum_{i=1}^{n_0} 2^{-i} \text{co} F_i$ is empty, there exists $v \in B_X(0, r)$ so that

$$(4) \quad \text{dist}(v, Q_0) > r/2.$$

To see this choose a point $y \in B(0, r/4) \setminus Q_0$ and x^* in the unit sphere of X^* which separates y from Q_0 , namely $r/4 \geq \langle x^*, y \rangle \geq \langle x^*, u \rangle$ for any $u \in Q_0$. Choose $x \in B(0, r)$ so that $\langle x^*, x \rangle > 3r/4$. Then $v = x$ satisfies the required inequality. From (3) and (4) follows that

$$\text{dist}(v, Q + \sum_{i=1}^{\infty} 2^{-i} F_i) \geq r/4,$$

which is a contradiction. $\square$

With the hypothesis above, observe that if $T : X \rightarrow Y$ is a surjective linear operator with finite-dimensional kernel F , then $T(Q)$ is a bounded, closed and convex set with empty interior that is also spanning; this is because $T^{-1}(T(X)) = Q + F$ is a first category set.

In connection with the next theorem observe that the positive cone of ℓ_2 is a closed convex set with empty interior which contains a translate of any finite subset F of ℓ_2 . (Indeed, if for $x = \{x_n\} \in \ell_2$ we define $x^- = \{x_n^-\}$ so that $x_n^- = -x_n$ if $x_n < 0$ and $x_n^- = 0$ otherwise, then the set $F + \sum_{x \in F} x^-$ is contained in the positive cone.) However, as we will see later, because ℓ_2 is reflexive it does not contain a closed convex set with empty interior containing a translate of every compact set. Hence the boundedness hypothesis in (iv) of the next theorem is needed.

Theorem 3. *Let X be a Banach space. Then the following are equivalent:*

- (i) *there exists a convex and closed subset Q of X with empty interior which contains translates of all compact sets in X ; i.e. whenever K is a compact subset of X there exists $x_K \in X$ so that $K + x_K \subset Q$;*
- (ii) *there exists a convex and closed subset P of X with empty interior such that if K is a compact subset of the unit ball of X , then there exists $x_K \in X$ so that $K + x_K \subset P$;*
- (iii) *there exists a convex, closed and bounded subset C of X with empty interior such that if K is a compact subset of the unit ball of X , then there exists $x_K \in X$ so that $K + x_K \subset C$; and*
- (iv) *there exist a dense subset E of the unit ball of X and a convex, closed and bounded subset D of X with empty interior so that whenever F is a finite set contained in E , there exists $x_F \in X$ so that $F + x_F \subset D$.*

Proof. Clearly (i) implies (ii). To prove that (ii) implies (iii), it is enough to show that there exists $1 \geq r > 0$ and $c > 0$ so that for any compact set $K \subset \bar{B}(0, r)$ there exist $z_K \in B(0, c)$ so that $K + z_K \subset P$, for then we may define

$$C := \frac{1}{r} (P \cap \bar{B}(0, r + c)).$$

For a contradiction, suppose that for every $n \in N$ there exists a compact set $K_n \subset \bar{B}(0, 1/n)$ so that

$$(5) \quad \text{if } K_n + x \subset P, \text{ then } \|x\| \geq n.$$

Define

$$K := \bigcup_{n=1}^{\infty} K_n \cup \{0\}.$$

The set K is a compact subset of the unit ball, hence there exists $z \in X$ such that $K + z \subset P$. Because $K_n \subset K$ for $n \in N$, we have $\|z\| \geq n$ for all n , which is nonsense.

Let us show now that (iii) implies (i). We can suppose that zero is not contained in C and define

$$Q := \bigcup_{\lambda \geq 0} \lambda C.$$

The set Q is convex and contains translates of all compact sets in X . To show that Q is closed, let $z \in X$, $x_n \in C$ and $\lambda_n \geq 0$ such that $\lim_{n \rightarrow \infty} \lambda_n x_n = z$ be given. Because the sequence $\{x_n\}$ is bounded away from zero, the sequence $\{\lambda_n\}$ is bounded, and consequently it has a converging subsequence $\lambda_{n_k} \rightarrow \lambda \geq 0$. If $\lambda = 0$, then from the boundedness of C it follows that $z = 0 \in Q$. Otherwise the sequence $\{x_{n_k}\}$ converges to z/λ . Because C is closed we get that $z = \lim_{k \rightarrow \infty} \lambda_{n_k} x_{n_k} = z \in \lambda C$. Finally, let us show that the set Q has empty interior. Choose some $z \in C$. The set $\tilde{C} := C + [-z, 0]$ is closed and convex, and because C is spanning $\tilde{C}$ has empty interior by Lemma 2. Since

$$Q = \bigcup_{\lambda \geq 0} \lambda C \subset \bigcup_{n \in N} n \tilde{C},$$

it follows from the Baire theorem that the interior of Q is empty.

Clearly (iii) implies (iv), so let us show that the opposite implication also holds. Let K be a compact subset of $B_X(0, 2^{-1})$. We will show that K can be translated into D . Then $C := 2D$ will satisfy (iii). Let $\{F_n\}$ be a sequence of finite sets in E so that (1) holds. Choose $z_n \in X$ so that $z_n + F_n \subset D$. Because D is bounded, the sequence $\{z_n\}$ is bounded. If we define $z := \sum_{n=1}^{\infty} (1/2^n) z_n$, we get

$$z + K \subset z + \text{cl} \sum_{n=1}^{\infty} 2^{-n} F_n \subset \text{cl} \left(\sum_{n=1}^{\infty} 2^{-n} z_n + 2^{-n} F_n \right) \subset D,$$

where the last inclusion follows from the fact that D is convex and closed. $\square$

It should be remarked here that from the proof of equivalence of (i) and (iii) of the previous theorem it follows that if a Banach space X contains a closed and convex set with empty interior containing the translates of all compacts, then X contains a closed and convex cone Q with empty interior which contains the translates of all compacts.

Corollary 4. *Let Z be a Banach space and Y be a separable subspace of Z . Let Z contain a convex closed set Q with empty interior which contains translates of all compact sets in Z . Then there exist a closed, separable and linear subspace X of Z containing Y and a convex closed subset P of X with empty interior which contains translates of all compact sets in X .*

Proof. By Theorem 3 there exists a convex closed bounded subset C of Z with empty interior which contains translates of all compact subsets of B_Z . Using induction we construct an increasing sequence $\{X_n\}$ of closed separable subspaces of Z . Define $X_1 := Y$ and choose a dense countable subset S_1 of the unit ball of X_1 . Choose a countable set $T_1 \subset Z$ such that whenever F is a finite subset of S_1 there exists $v \in T_1$ for which $v + F \subset C$. Choose a countable set $C_1 \subset Z \setminus C$ such that $\overline{C_1} \supset C \cap X_1$. Suppose X_n, S_n, T_n and C_n for some $n \in N$ have been already constructed. Define

$$X_{n+1} := \overline{\text{span}}(X_n \cup T_n \cup C_n),$$

and choose a countable dense subset $S_{n+1} \supset S_n$ of the unit ball of X_{n+1} . Choose a countable set $T_{n+1} \subset Z$ such that whenever F is a finite subset of S_{n+1} there exists $v \in T_{n+1}$ for which $v + F \subset C$. Choose a countable set $C_{n+1} \subset Z \setminus C$ such that $\overline{C_{n+1}} \supset C \cap X_{n+1}$. Define

$$X := \bigcup_{n=1}^{\infty} X_n \quad \text{and} \quad D := \overline{\bigcup_{n=1}^{\infty} (X_n \cap C)}.$$

The set $E := \bigcup_{n=1}^{\infty} S_n$ is dense in $\bar{B}_X$ and from the construction it follows that any finite set contained in E can be translated into D . The set D is closed and convex, and it has empty interior because $\bigcup_{n=1}^{\infty} C_n \subset X \setminus D$ and $\overline{\bigcup_{n=1}^{\infty} C_n} \supset D$. An application of Theorem 3 completes the proof. $\square$

The following lemma is essentially due to James [4].

Lemma 5. *Let X be a nonreflexive Banach space. Then there exists a sequence $\{x_n\}$ in the unit ball of X and $\varepsilon > 0$ so that for any finite-dimensional subspace Y of X there exists $n \in N$ so that*

$$\text{dist}(Y, \text{co}\{x_i\}_{i=n}^{\infty}) > \varepsilon.$$

Proof. The unit ball $\bar{B}_X$ of X is not weakly compact, therefore by the Gantmacher-Smulyan theorem [3] there exists a decreasing sequence $\{C_n\}$ of nonempty, closed and convex subsets of $\bar{B}_X$ such that $\bigcap_{n=1}^{\infty} C_n = \emptyset$. We will show that there exist $\varepsilon > 0$ and a decreasing sequence of convex nonempty sets $\{D_n\}$ so that $D_n \subset C_n$ for $n \in N$ and for any compact set $K \subset X$ there exists $m \in N$ such that

$$(K + B(0, \varepsilon)) \cap D_m = \emptyset.$$

Suppose for a contradiction that the required sequence $\{D_n\}$ does not exist. Let $C_{1,n} := C_n$ for $n \in N$. There exists a compact convex set K_1 so that

$$C_{1,n} \cap (K_1 + B(0, 2^{-1})) \neq \emptyset \quad \text{for } n \in N.$$

Let $C_{2,n} := C_{1,n} \cap (K_1 + B(0, 2^{-1}))$ for $n \in N$. In general, if the sequence $\{C_{k,n}\}$ and the compact convex set K_k have been already constructed, define

$$C_{k+1,n} := C_{k,n} \cap (K_k + B(0, 2^{-k})) \quad \text{for } n \in N,$$

and choose a compact convex set K_{k+1} so that

$$C_{k+1,n} \cap (K_{k+1} + B(0, 2^{-(k+1)})) \neq \emptyset \quad \text{for } n \in N.$$

Then $C_{k+1,n} \subset C_{k,n}$, and by induction $C_{k,n+1} \subset C_{k,n}$. In particular if we define $G_n := C_{n,n}$, then the sequence $\{G_n\}$ is decreasing, $G_n \subset C_n$ and

$$G_{n+1} \subset K_n + B(0, 2^{-n}).$$

Choose some $y_n \in G_n$. The sequence $\{y_n\}$ has a finite δ -net for any $\delta > 0$. Therefore it has a converging subsequence. The limit point of this subsequence is contained in $\bigcap_{n=1}^{\infty} C_n$, which is a contradiction. Now that we have shown the existence of the sequence $\{D_n\}$, to finish the proof simply choose any $x_n \in D_n$. $\square$

Theorem 6. *Let Z be a Banach space. The following are equivalent:*

- (i) Z is not reflexive;
- (ii) *there exist a nontrivial closed subspace X of Z and a convex closed subset Q of X with empty interior which contains translates of all compact sets in X , i.e. whenever K is a compact subset of X there exists $x_K \in X$ so that $K + x_K \subset Q$.*

Moreover, if Z is separable, then (ii) holds with $X = Z$.

Proof. To show that (i) implies (ii) choose any separable nonreflexive subspace X of Z ; such a space exists by the Eberlein-Smulyan theorem. If Z is separable let $X := Z$. Choose an increasing sequence $\{X_n\}$ of finite-dimensional subspaces of X so that $X = \overline{\bigcup_{n=1}^{\infty} X_n}$. Choose a sequence $\{x_n\}$ in the unit ball of X and $\varepsilon > 0$ as in Lemma 5. By passing to a subsequence of $\{x_n\}$ if necessary we may suppose that

$$(6) \quad \text{dist}(\text{span}(X_n \cup \{x_i\}_{i=1}^n), \text{co}\{x_i\}_{i=n+1}^{\infty}) > \varepsilon \quad \text{for } n \in N.$$

Put $K_n := X_n \cap B_Z$, and define

$$D := \overline{\text{co}} \bigcup_{i=1}^{\infty} (x_i + (\varepsilon/4)K_i).$$

The convex, closed and bounded set $\tilde{D} := (4/\varepsilon)D$ contains a translate of any finite subset of $\bar{B}_X \cap \bigcup_{n=1}^{\infty} X_n$. By Theorem 3, it only remains to show that the interior of $\tilde{D}$ is empty. For a contradiction, suppose that the interior of D is nonempty. Because $\text{co} \bigcup_{i=1}^{\infty} (x_i + (\varepsilon/4)K_i)$ is dense in D there exist $n \in N$, $\alpha_i \geq 0$ and $u_i \in (\varepsilon/4)K_i$, $i = 1, \dots, n$, so that $\sum_{i=1}^n \alpha_i = 1$ and the point $z := \sum_{i=1}^n \alpha_i(x_i + u_i)$ is contained in the interior of D . From (6) it follows that there exists a point x^* in the unit sphere of X^* so that

$$\begin{aligned} \langle x^*, x \rangle &= 0 & \text{for } x \in X_n, \\ \langle x^*, x \rangle &\leq -\varepsilon/2 & \text{for } x \in \text{co}\{x_i\}_{i=n+1}^{\infty}. \end{aligned}$$

Choose a point w in the unit sphere of X for which

$$\langle x^*, w \rangle \geq 1/2.$$

Since z is an interior point of D , there exists an $r > 0$ so that $z + rw \in D$. Consequently, there exist $m \in N$, $m > n$, $\beta_i \geq 0$ and $v_i \in (\varepsilon/4)K_i$, $i = 1, \dots, m$, so that $\sum_{i=1}^m \beta_i = 1$ and if we define $y := \sum_{i=1}^m \beta_i(x_i + v_i)$, then

$$\|z + rw - y\| < r/2.$$

From the definition of x^* it follows that

$$\begin{aligned} &\langle rw + z - y, x^* \rangle \\ &= r\langle w, x^* \rangle + \langle \sum_{i=1}^n \alpha_i(x_i + u_i) - \beta_i(x_i + v_i), x^* \rangle - \langle \sum_{i=n+1}^m \beta_i(x_i + v_i), x^* \rangle \\ &\geq r/2 + 0 - \sum_{i=n+1}^m \beta_i(\langle x_i, x^* \rangle + \langle v_i, x^* \rangle) \\ &\geq r/2 - \sum_{i=n+1}^m \beta_i(-\varepsilon/2 + \varepsilon/4) \\ &\geq r/2, \end{aligned}$$

which is a contradiction.

Now, let us prove that (ii) implies (i). By Corollary 4, we may suppose that X is separable. We will show that X is nonreflexive and therefore Z is also nonreflexive. For a contradiction suppose that X is reflexive. Choose a sequence $\{x_i\}_{i=1}^\infty \subset X$ that is dense in the unit sphere of X . Denote

$$K_n := \text{span}\{x_i\}_{i=1}^n \cap \bar{B}_X.$$

Clearly $\{K_n\}$ is an increasing sequence of compact subsets of the unit ball of X for which

$$(7) \quad \overline{\bigcup_{n=1}^\infty K_n} = \bar{B}_X.$$

By Theorem 3 there exists a closed, convex and bounded subset C of X with empty interior which contains translates of all compact subsets of the unit ball of X . For $n \in \mathbb{N}$ choose $z_n \in X$ so that $z_n + K_n \subset C$. The sequence $\{z_n\}$ is bounded, therefore it has a weakly converging subsequence $\{z_{n_k}\}$. Denote $z := w\text{-}\lim_{k \rightarrow \infty} z_{n_k}$. Because the set C is convex and closed, it is also weakly closed. Consequently, because the sets K_n are increasing, if there exists a $k \in \mathbb{N}$ so that $y \in K_{n_k}$, then $y + z \in C$. Hence,

$$(8) \quad z + \bar{B}_X = z + \overline{\bigcup_{k=1}^\infty K_{n_k}} \subset C,$$

which, of course, means that the interior of C is nonempty, which is a contradiction. $\square$

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