ON THE DENSITY OF PROPER EFFICIENT POINTS

FU WANTAO

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ABSTRACT. In this paper, our aim is to discuss the density of proper efficient points. As an interesting application of the results in this paper, we want to prove a density theorem of Arrow, Barankin, and Blackwell.

In [1], Luc introduced a new concept of the proper efficient point for a set. Using some results of recession cone, Luc established efficiency conditions, especially proper efficiency and domination properties ([1, 2]). The present paper is devoted to the study of the density of proper efficient points. In detail, the set of proper efficient points for a set is dense in the set of efficient points. As an interesting application of the results in this paper, we prove a density theorem of Arrow, Barankin, and Blackwell ([3, 4]).

First let us recall some notations:

Throughout the paper, E is a separated locally convex topological linear space and E^* its topological dual. $U(0)$ denotes the family of balanced open convex neighbourhoods of the origin in E . For $A \subset E$, $\text{cone}(A)$, $\text{cl}(A)$, and $\text{int}(A)$ denote the generated cone, the closure, and the interior of A , respectively.

Let $C \subset E$ be a convex cone, and let A be a nonempty subset of E . We say that $x \in A$ is an efficient point of A with respect to C if there exists $y \in A$, such that $y \in x - C$; then $y \in x + C$. Equivalently, $(x - C) \cap A \subset x + C$. If the C is pointed (that is, $C \cap (-C) = \{0\}$), then $x \in A$ is an efficient point iff

$$(x - C) \cap A = \{x\}.$$

We denote by $E(A, C)$ the set of all efficient points of A (with respect to C). We say that $x \in A$ is a proper efficient point of A with respect to C if there exists a closed convex cone $K \neq E$ such that $C \setminus \{0\} \subset \text{int}(K)$ and $x \in E(A, K)$.

The set of proper efficient points of A is denoted by $\text{Prop } E(A, C)$. It is obvious that the set of proper efficient points of A is contained in the set of efficient points,

$$\text{Prop } E(A, C) \subset E(A, C),$$

but the converse is not generally true.

If C is a convex cone, the convex set $B \subset C$ is said to be a base of C if

$$0 \notin \text{cl}(B) \quad \text{and} \quad C = \text{cone}(B) = \bigcup \{tB : t \geq 0\} = \{tb : t \geq 0, b \in B\}.$$

A cone with base must be pointed.

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For the cone C with base B , we define the “expansion” cone as below.

According to $0 \notin \text{cl}(B)$, there exists a balanced convex open neighbourhood $U^* \in U(0)$ such that

$$U^* \cap B = \emptyset.$$

Let $N(0) = \{U \subset U^* : U \in U(0)\}$, and for $U \in N(0)$, let

$$C_U = \text{cl}(\text{cone}(B + U)).$$

C_U is said to be an expansion cone of C . It is clear that the expansion cone C_U is a closed convex cone.

Lemma 1. *For any $U \in N(0)$, we have that*

$$C \setminus \{0\} \subset \text{int}(C_U) \quad \text{and} \quad 0 \notin \text{int}(C_U).$$

Proof. Let $x \in C \setminus \{0\}$. There exists $b \in B, t > 0$, such that $x = tb$, so

$$x + tU = tb + tU \subset t(B + U) \subset \text{cone}(B + U) \subset C_U.$$

Therefore, $x \in \text{int}(C_U)$.

Assume that $0 \in \text{int}(C_U)$, for some $U \in N(0)$; then $C_U = E$. Choose any $b \in B$. By $-b \in C_U$, there is some net $\{\mu_\tau(b_\tau + u_\tau) : \tau \in \Lambda\}, \mu_\tau > 0, b_\tau \in B, u_\tau \in U$, such that $\mu_\tau(b_\tau + u_\tau) \rightarrow -b$, consequently $\mu_\tau(b_\tau + u_\tau) + b \rightarrow 0$. Since U is a neighbourhood of the zero, there exists $\tau_0 \in \Lambda$ such that

$$\mu_{\tau_0}(b_{\tau_0} + u_{\tau_0}) + b \in U,$$

or

$$\mu_{\tau_0}(b_{\tau_0} + u_{\tau_0}) + b = v_{\tau_0}, \quad v_{\tau_0} \in U,$$

and consequently

$$\frac{\mu_{\tau_0}}{\mu_{\tau_0} + 1}b_{\tau_0} + \frac{1}{\mu_{\tau_0} + 1}b = \frac{1}{\mu_{\tau_0} + 1}v_{\tau_0} + \frac{\mu_{\tau_0}}{\mu_{\tau_0} + 1}(-u_{\tau_0}) := w_{\tau_0}.$$

This implies $w_{\tau_0} \in B \cap U$ (since U is balanced and convex, and B is convex), which contradicts that $B \cap U = \emptyset$. This contradiction shows that $0 \notin \text{int}(C_U)$, for all $U \in N(0)$. The proof of Lemma 1 is complete.

Theorem 1. *Assume that C is a closed convex cone with base B . Then for any compact set $A \subset E$, the set of proper efficient points of A is dense in the set of efficient points of A ,*

$$E(A, C) \subset \text{cl}(\text{Prop } E(A, C)).$$

Proof. Let $A \neq \emptyset$ be a compact set, and $x \in E(A, C)$. Without loss of generality, we may assume that $x = 0$, thus $(0 - C) \cap A = \{0\}$. We must show that there exists a net $\{x_\tau\} \subset \text{Prop } E(A, C)$ such that $x_\tau \rightarrow 0$.

For any $U \in N(0)$, assume that C_U is the expansion cone, and let $A_U = (0 - C_U) \cap A$. Since A_U is compact, by using the usual existence theorem of efficient points (see [7, p. 140, Theorem 6.3]), there exists an efficient point of A_U with respect to C_U . Let $x_U \in E(A_U, C_U)$. Since A_U is a section of A at 0, $x_U \in E(A, C_U)$. Of course C_U is a closed convex cone with $C \setminus \{0\} \subset \text{int}(C_U)$, so

$$x_U \in \text{Prop } E(A, C) \quad \text{for } U \in N(0).$$

Since A is compact and $\{x_U : U \in N(0)\} \subset A$, without loss of generality, we may assume that $x_U \rightarrow x \in A$.

Now we have to prove that $x = 0$.

Since $x_U \in A_U = (0 - C_U) \cap A$, $-x_U \in C_U$. Hence there exists a net $t_\tau(b_\tau + u_\tau) \rightarrow -x_U$, where $t_\tau \geq 0$, $b_\tau \in B$, $u_\tau \in U$. Considering $-x_U + U$ is a neighbourhood of $-x_U$, there exists some index τ_U such that

$$t_{\tau_U}(b_{\tau_U} + u_{\tau_U}) \in -x_U + U$$

or

$$(1) \quad t_{\tau_U}(b_{\tau_U} + u_{\tau_U}) = -x_U + y_U, \quad y_U \in U.$$

The number set $\{t_{\tau_U} : U \in N(0)\}$ must be bounded. Otherwise, we may assume that $t_{\tau_U} \rightarrow +\infty$. Obviously, the two nets $\{u_{\tau_U} : U \in N(0)\}$ and $\{y_U : U \in N(0)\}$ converge to 0. From (1) follows that

$$b_{\tau_U} = \frac{-x_U + y_U}{t_{\tau_U}} - u_{\tau_U} \rightarrow 0;$$

this implies $0 \in \text{cl}(B)$, which is a contradiction, since $0 \notin \text{cl}(B)$.

Since $\{t_{\tau_U}\}$ is a bounded number set, we may assume without loss of generality that $t_{\tau_U} \rightarrow t \geq 0$. By (1)

$$t_{\tau_U}b_{\tau_U} = (-x_U + y_U) - t_{\tau_U}u_{\tau_U} = -x_U + (y_U - t_{\tau_U}u_{\tau_U}) \rightarrow -x.$$

Considering $\{t_{\tau_U}b_{\tau_U}\} \subset C$, we conclude that $-x \in C$, or $x \in -C$. Since $0 \in E(A, C)$, and C is pointed, so $x \in (-C) \cap A = (0 - C) \cap A = \{0\}$, therefore, $x = 0$, consequently $x_U \rightarrow 0$. This completes the proof.

The result stated in Theorem 1 can be extended to weakly compact sets. Note that the symbol “ $\rightharpoonup$ ” denotes weak convergence.

Theorem 2. *Let C be a closed convex cone with bounded base B . Then for any weakly compact set $A \subset E$, the set of proper efficient points of A is dense in the set of efficient points of A ,*

$$E(A, C) \subset \text{cl}(\text{Prop } E(A, C)).$$

Proof. Let us replace strong convergence by weak convergence, and note that for a convex set $\text{w-cl}(\cdot) = \text{cl}(\cdot)$. By using the same proof of Theorem 1, we can conclude that there exists a net $\{x_U : U \in N(0)\} \subset \text{Prop } E(A, C)$ such that $x_U \rightharpoonup 0 \in E(A, C)$.

Now, we must prove that $x_U \rightarrow 0$. According to (1) in Theorem 1,

$$t_{\tau_U}(b_{\tau_U} + u_{\tau_U}) = -x_U + y_U \quad \text{for } U \in N(0),$$

where $u_{\tau_U} \rightarrow 0$, $y_U \rightarrow 0$, and $t_{\tau_U} \rightarrow t \geq 0$.

If $t > 0$, then

$$b_{\tau_U} = \frac{-x_U + y_U}{t_{\tau_U}} - u_{\tau_U} \rightharpoonup 0.$$

This implies $0 \in \text{w-cl}(B) = \text{cl}(B)$, which is a contradiction. Therefore $t = 0$, consequently $t_{\tau_U} \rightarrow 0$.

Since B is bounded, $\{b_{\tau_U}\} \subset B$ is bounded. Therefore,

$$x_U = y_U - t_{\tau_U}(b_{\tau_U} + u_{\tau_U}) \rightarrow 0.$$

The proof is complete.

As an interesting application of the above theorems, we want to prove the Arrow-Barankin-Blackwell theorem. For convenience, we introduce first some concepts and definitions.

Let C be a closed convex cone, and let

$$C^\sharp = \{f \in E^* : f(x) > 0 \text{ for all } x \in C \setminus \{0\}\}.$$

An element $f \in E^*$ is said to be strictly positive if $f \in C^\sharp$. Let $A \neq \emptyset$. $\bar{x} \in A$ is said to be a positively proper efficient point of A if there exists some strictly positive functional $f \in C^\sharp$ such that

$$f(\bar{x}) \leq f(x) \quad \text{for all } x \in A,$$

i.e.

$$f(\bar{x}) = \min\{f(x) : x \in A\}.$$

We denote by $\text{Ps } E(A, C)$ the set of those elements $\bar{x} \in A$ satisfying the above condition. Obviously, for any subset A , one has that

$$\text{Ps } E(A, C) \subset E(A, C).$$

The converse is not true, but we can very simply show the following conclusions.

Theorem 3 (the Arrow-Barankin-Blackwell theorem [3–6]). (i) *Let $C \subset E$ be a closed convex cone with base B . Then for any compact convex set $A \subset E$, the positively proper efficient point set of A is dense in the efficient point set,*

$$E(A, C) \subset \text{cl}(\text{Ps } E(A, C)).$$

(ii) *Let $C \subset E$ be a closed convex cone with bounded base B . Then for any weakly compact convex set A , the above result is true.*

Proof. It is sufficient to show only that

$$\text{Prop } E(A, C) \subset \text{Ps } E(A, C).$$

Indeed, let $\bar{x} \in \text{Prop } E(A, C)$. By hypothesis, there exists a closed convex cone K such that

$$C \setminus \{0\} \subset \text{int}(K) \quad \text{and} \quad \bar{x} \in E(A, K).$$

Since $\bar{x} \in E(A, K)$, $(\bar{x} - K) \cap A \subset \bar{x} + K$. Consequently, we get $(\bar{x} - K \setminus l(K)) \cap A = \emptyset$, where $l(K) = K \cap (-K)$. Choose any $x \in \text{int}(K)$. If $x \in l(K)$, then $x \in -K$, or $-x \in K$. This implies that $0 = \frac{1}{2}x + \frac{1}{2}(-x) \in \text{int}(K)$ (since K is convex), which contradicts the assumption that $0 \notin \text{int}(K)$. Therefore $x \in \text{int}(K \setminus l(K))$. So, we obtain $\emptyset \neq \text{int}(K) \subset \text{int}(K \setminus l(K))$. Using a separating theorem, there exists $0 \neq f \in E^*$ and a real t such that

$$\begin{aligned} \sup f(\bar{x} - K \setminus l(K)) &\leq t \leq \inf f(A), \\ \sup f(\bar{x} - \text{int}(K \setminus l(K))) &< t \leq \inf f(A). \end{aligned}$$

As $\bar{x} \in A$ and $C \setminus \{0\} \subset \text{int}(K) \subset \text{int}(K \setminus l(K))$, the second inequality implies $f \in C^\sharp$. Furthermore, notice that for any $x \in K$, $\exists \{x_n\} \subset K \setminus l(K)$, such that $x_n \rightarrow x$, therefore, from the first inequality follows that $\sup f(\bar{x} - K) \leq t \leq \inf f(A)$. By $0 \in K$, we have $f(\bar{x}) \leq f(x)$ for all $x \in A$, this is $\bar{x} \in \text{Ps } E(A, C)$. So

$$\text{Prop } E(A, C) \subset \text{Ps } E(A, C);$$

this relation implies $\text{cl}(\text{Prop } E(A, C)) \subset \text{cl}(\text{Ps } E(A, C))$.

Now, using Theorem 1 (respectively, Theorem 2), we obtain

$$E(A, C) \subset \text{cl}(\text{Prop } E(A, C)) \subset \text{cl}(\text{Ps } E(A, C)).$$

The proof is complete.

CONCLUDING REMARKS

Borwein in [4] proved that every efficient point of a weakly compact convex set in a finite-dimensional space is a limit of Borwein's properly efficient points. Using the concept of an approximating family of cones, and relaxing the convexity assumption imposed upon the objective set, Helbig [10] proved the above fact, too. In 1989, Sterna-Karwat discussed the existence of approximating families of cones in normed spaces, and using this concept, she showed that every efficient point of a weakly compact set in a normed space can be approximated by properly efficient points. Clearly, the approximating family of a cone cannot be extended onto topological vector spaces because this notion exists not in topological vector spaces (see [13]). In this paper, in order to derive the density theorem of properly efficient points in Luc's sense which is a generalization of Henig's properly efficient point in normed spaces, we consider a generalised concept of approximation families of cones, in which the "pointed" property is not required.

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