CHARACTERIZATION OF THE FOURIER SERIES OF A DISTRIBUTION HAVING A VALUE AT A POINT

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(Communicated by J. Marshall Ash)

ABSTRACT. Let f be a periodic distribution of period 2π . Let $\sum_{n=-\infty}^{\infty} a_n e^{in\theta}$ be its Fourier series. We show that the distributional point value $f(\theta_0)$ exists and equals γ if and only if the partial sums $\sum_{-x \leq n \leq ax} a_n e^{in\theta_0}$ converge to γ in the Cesàro sense as $x \rightarrow \infty$ for each $a > 0$.

1. INTRODUCTION

The study of the relationship between the local behavior of a periodic function at a point and the convergence and summability of the corresponding Fourier series has been one of the main concerns of classical harmonic analysis, a subject with many years of history.

In particular, the characterization of the Fourier series having a “value” at a point is a very interesting problem, whose solution depends on the notion of value used. Among the various useful notions of value we could mention the value of a continuous function at a point, the measure-theoretical approximate value of an integrable function, or the distributional point value of a generalized function.

In this article we shall be concerned with the characterization of the Fourier series of generalized functions having a distributional point value. The notion of distributional point value was introduced by Łojasiewicz [8] and corresponds, roughly, to the existence of value “on the average”. If $f \in \mathcal{D}'(\mathbb{R})$ is a distribution and $x_0 \in \mathbb{R}$, we say that f has the distributional value γ at $x = x_0$, and write $f(x_0) = \gamma$ in $\mathcal{D}'$ if for each $\phi \in \mathcal{D}$ we have $\lim_{\varepsilon \rightarrow 0} \langle f(x_0 + \varepsilon x), \phi(x) \rangle = \gamma \int_{-\infty}^{\infty} \phi(x) dx$ (if f is given by an integral and $\int_{-\infty}^{\infty} \phi(x) dx = 1$ this means that the “average” $\int_{-\infty}^{\infty} f(x_0 + \varepsilon x) \phi(x) dx$ tends to γ). It can be shown that $f(x_0) = \gamma$ in $\mathcal{D}'$ if and only if there exists a primitive of order k of f , i.e., $F^{(k)} = f$, which is continuous in a neighborhood of $x = x_0$ and satisfies $\lim_{x \rightarrow x_0} k! (x - x_0)^{-k} F(x) = \gamma$. This corresponds to the notion of generalized derivatives (see [13] and the references therein) used in the theory of trigonometric series well before the definition of Łojasiewicz and even before the introduction of the notion of distributions by Schwartz [10].

It turns out that the existence of distributional point values is equivalent to the existence *in the Cesàro sense* of the limits of certain partial sums of the corresponding Fourier series. Convergence in the Cesàro sense is also a kind of “convergence

on the average" and is defined as follows [6]: if f is an integrable function defined for $x \geq 0$ we say that $F(x)$ tends to L in the $(C, 1)$ sense as $x \rightarrow +\infty$, and write $\lim_{x \rightarrow +\infty} F(x) = L$ $(C, 1)$ if $\lim_{x \rightarrow +\infty} \frac{1}{x} \int_0^x F(t) dt = L$. Convergence in the (C, k) sense, $k = 2, 3, \dots$, is defined recursively by $\lim_{x \rightarrow +\infty} F(x) = L$ (C, k) if $\lim_{x \rightarrow +\infty} \frac{1}{x} \int_0^x F(t) dt = L$ $(C, k-1)$. We say that $F(x)$ converges to L in the Cesàro sense and write $\lim_{x \rightarrow +\infty} F(x) = L$ (C) if $\lim_{x \rightarrow +\infty} F(x) = L$ (C, k) for some k .

The Cesàro convergence of a sequence $\{x_n\}$ can be defined similarly and it is equivalent to the Cesàro convergence of the function F defined by $F(t) = x_{[t]}$. In particular a series $\sum_{n=1}^{\infty} a_n$ is Cesàro summable to the sum S , written as $\sum_{n=1}^{\infty} a_n = S$ (C) , if $\lim_{n \rightarrow \infty} \sum_{j=1}^n a_j = S$ (C) . A series $\sum_{n=-\infty}^{\infty} a_n$ is (C) summable to S if both series $\sum_{n=0}^{\infty} a_n$ and $\sum_{n=1}^{\infty} a_{-n}$ are (C) summable to S_1 and S_2 , respectively, with $S = S_1 + S_2$. A series $\sum_{n=-\infty}^{\infty} a_n$ is principal value Cesàro summable to S , written as p.v. $\sum_{n=-\infty}^{\infty} a_n = S$ (C) , if $\lim_{n \rightarrow \infty} \sum_{j=-n}^n a_j = S$ (C) .

Now let $f \in \mathcal{S}'$ be a periodic distribution of period 2π and let $\sum_{n=-\infty}^{\infty} a_n e^{in\theta}$ be the associated Fourier series. Let $\theta_0 \in \mathbb{R}$. Then the following two results are known [3, 12]:

(1) If $\sum_{n=-\infty}^{\infty} a_n e^{in\theta_0} = \gamma$ (C) , i.e., if the partial sums $\sum_{n=-N}^M a_n e^{in\theta_0}$ tend to γ in the Cesàro sense as N and M tend to infinity independently, then $f(\theta_0) = \gamma$ in $\mathcal{D}'$. The converse does not hold ($f(\theta) = \sum_{n=2}^{\infty} \frac{e^{in\theta}}{n \log n}$, at $\theta = 0$, is an example).

(2) If $f(\theta_0) = \gamma$ in $\mathcal{D}'$, then p.v. $\sum_{n=-\infty}^{\infty} a_n e^{in\theta_0} = \gamma$ (C) , i.e.,

$$\lim_{N \rightarrow \infty} \sum_{n=-N}^N a_n e^{in\theta_0} = \gamma \quad (C).$$

The converse does not hold ($f(\theta) = \sum_{n=1}^{\infty} n \sin n\theta$, at $\theta = 0$, is an example).

Here we prove the following

Theorem. Let $f \in \mathcal{S}'$ be a periodic distribution of period 2π and let $\sum_{n=-\infty}^{\infty} a_n e^{in\theta}$ be its Fourier series. Let $\theta_0 \in \mathbb{R}$. Then

$$(1.1) \quad f(\theta_0) = \gamma, \quad \text{in } \mathcal{D}',$$

if and only if there exists k such that

$$(1.2) \quad \lim_{x \rightarrow +\infty} \sum_{-x \leq n \leq ax} a_n e^{in\theta_0} = \gamma, \quad (C, k)$$

for each $a > 0$.

Our approach is based on the theory of distributional asymptotic expansions [4, 5, 11] and is inspired by the work of Ramanujan [9] who was one of the first to study a sequence $\{a_n\}$ by studying the asymptotic behavior of series of the type $\sum_{n=1}^{\infty} a_n \phi(n\varepsilon)$, as $\varepsilon \rightarrow 0^+$, for smooth ϕ . See also [1, 2]. Here we study the Fourier series $\sum_{n=-\infty}^{\infty} a_n e^{in\theta_0}$ by analysing the behavior as $\varepsilon \rightarrow 0$ of the series $\sum_{n=-\infty}^{\infty} a_n e^{in\theta_0} \phi(n\varepsilon)$ for $\phi \in \mathcal{S}$.

Notice that we use the standard notation concerning spaces of distributions and test functions [7, 10].

2. ASYMPTOTICALLY HOMOGENEOUS FUNCTIONS

In this section we define and give the basic properties of the asymptotically homogeneous functions of degree 0. These functions play a central role in our analysis.

Lemma 1. *Let τ be a real-valued continuous function defined in an interval of the form $[A, \infty)$ for some $A \in \mathbb{R}$. Suppose*

$$(2.1) \quad \tau(ax) = a^\mu \tau(x) + o(1), \quad \text{as } x \rightarrow +\infty,$$

for each $a > 0$. If $\mu < 0$, then

$$(2.2) \quad \tau(x) = o(1), \quad \text{as } x \rightarrow +\infty.$$

Proof. Let $\varepsilon > 0$. Let $x_0 > 0$ be such that $|\tau(2x) - 2^\mu \tau(x)| \leq \varepsilon$ for $x > x_0$. Let $M = \max\{|\tau(x)| : x_0 \leq x \leq 2x_0\}$. An inductive argument shows that if $x \in [2^n x_0, 2^{n+1} x_0]$, $n = 0, 1, 2, \dots$, then $|\tau(x)| \leq 2^{n\mu} M + \sum_{j=1}^{n-1} 2^{j\mu} \varepsilon$. Therefore, $\lim_{x \rightarrow +\infty} |\tau(x)| \leq \frac{2^\mu}{1-2^\mu} \varepsilon$ and (2.2) follows. $\square$

The lemma does not hold if $\mu = 0$. Indeed, there are functions like $\ln(\ln x)$, $|\ln x|^\alpha$, $\alpha < 1$, or $\cos \sqrt{|\ln x|}$ that satisfy $\tau(ax) = \tau(x) + o(1)$, as $x \rightarrow +\infty$, for each $a > 0$, but which do not tend to zero at infinity.

Definition. Let τ be a continuous function defined in an interval of the form $[A, \infty)$ for some $A \in \mathbb{R}$. We say that τ is asymptotically homogeneous of degree 0 if for each $a > 0$ we have

$$(2.3) \quad \tau(ax) = \tau(x) + o(1), \quad \text{as } x \rightarrow +\infty.$$

The asymptotically homogeneous functions of degree 0 are related to the slowly oscillating functions [5], also known as regularly varying functions of order 0. These are positive functions that satisfy $\rho(ax) = \rho(x) + o(\rho(x))$, as $x \rightarrow \infty$, for each $a > 0$. However, the two concepts are different: $\ln x$ is slowly oscillating but not asymptotically homogeneous of degree 0 while $\cos \sqrt{|\ln x|}$ is asymptotically homogeneous of degree 0 but not slowly oscillating.

Observe that the argument of the proof of Lemma 1 shows that if τ is an asymptotically homogeneous function of order 0, then $\tau(x) = o(\ln x)$, as $x \rightarrow +\infty$.

Notice also that we did not ask for any uniform behavior with respect to a in (2.3). The fact that (2.3) holds uniformly on $a \in [A, B]$ if $[A, B] \subseteq (0, \infty)$ follows from the definition, as we shall see.

Lemma 2. *Let τ be an asymptotically homogeneous function of degree 0, continuous in $[0, \infty)$. Let H be the Heaviside function. Then*

$$(2.4) \quad \tau(\lambda x)H(x) = \tau(\lambda)H(x) + o(1), \quad \text{as } \lambda \rightarrow \infty, \text{ in } \mathcal{S}'.$$

If $[A, B] \subseteq (0, \infty)$, then (2.3) holds uniformly for $a \in [A, B]$.

Proof. Suppose first that τ is bounded in $[0, \infty)$. Then if $\phi \in \mathcal{S}'$ we can apply the Lebesgue dominated convergence theorem to obtain

$$\lim_{\lambda \rightarrow \infty} \int_0^\infty (\tau(\lambda x) - \tau(\lambda)) \phi(x) dx = 0.$$

This is (2.4).

The uniform convergence follows from the distributional formula (2.4). Indeed, weak convergence in $\mathcal{S}'$ implies strong convergence [10]. Thus if K is a

compact subset of $\mathcal{S}$, then $\langle \tau(\lambda x)H(x) - \tau(\lambda)H(x), \phi(x) \rangle = o(1)$ uniformly for $\phi \in K$. Let $\phi_0 \in \mathcal{S}$ be a function that satisfies $\int_0^\infty \phi_0(x) dx = 1$. Then if $[A, B] \subseteq (0, \infty)$ the set $K = \{a^{-1}\phi_0(a^{-1}x) : a \in [A, B]\}$ is a compact set of $\mathcal{S}$. Then $\langle \tau(\lambda x)H(x), \phi_0(x) \rangle = \tau(\lambda) + o(1)$ as $\lambda \rightarrow \infty$, and so $\tau(\lambda a) = \langle \tau(\lambda a x), \phi_0(x) \rangle + o(1) = \langle \tau(\lambda x), a^{-1}\phi_0(a^{-1}x) \rangle + o(1) = \tau(\lambda) + o(1)$ uniformly for $a \in [A, B]$.

Let us now return to the general case when τ is not necessarily bounded. We shall first show that if $[A, B] \subseteq (0, \infty)$, then $\tau(ax) = \tau(x) + o(1)$, as $x \rightarrow +\infty$, uniformly on $a \in [A, B]$. Observe first that the functions $\cos \tau(x)$ and $\sin \tau(x)$ are bounded asymptotically homogeneous functions. By what we have already proven it follows that if $[A, B] \subseteq (0, \infty)$, then $e^{i\tau(ax)} = e^{i\tau(x)} + o(1)$, as $x \rightarrow \infty$, uniformly on $a \in [A, B]$. Let $\varepsilon > 0$. Suppose $\varepsilon < \pi$ and $A < 1 < B$. There exists $x_0 > 0$ such that $|e^{i\tau(ax)} - e^{i\tau(x)}| \leq |1 - e^{i\varepsilon}|$ for $x \geq x_0$ and $a \in [A, B]$. For each $x \geq x_0$ the set $\{\tau(ax) : A \leq a \leq B\}$ is a connected set contained in $\bigcup_{n=-\infty}^\infty [\tau(x) - \varepsilon + 2n\pi, \tau(x) + \varepsilon + 2n\pi]$ and it follows that it is contained in the component that contains $\tau(x)$; that is, $|\tau(ax) - \tau(x)| \leq \varepsilon$, for $a \in [A, B]$.

To prove (2.4) it would be enough to prove that there are constants A_0, A_1 such that $|\tau(\lambda x) - \tau(\lambda)| \leq A_0 |\ln x| + A_1$ for $x > 0$ and $\lambda > \lambda_0$ for then the dominated convergence theorem could be invoked again.

Let $\lambda_0 > 0$ be such that $|\tau(\lambda x) - \tau(\lambda)| \leq 1$ if $\lambda \geq \lambda_0$ and $x \in [1/2, 2]$. Then it follows by induction that $|\tau(\lambda x) - \tau(\lambda)| \leq n + 1$ if $x \in [2^n, 2^{n+1}]$, $\lambda \geq \lambda_0$, $n = 0, 1, 2, \dots$. Therefore $|\tau(\lambda x) - \tau(\lambda)| \leq \frac{|\ln x|}{\ln 2} + 1$ if $x \geq 1$, $\lambda \geq \lambda_0$. Proceeding similarly, if $x \in [\frac{1}{2^{n+1}}, \frac{1}{2^n}]$, $\lambda \geq 2^n \lambda_0$, $n = 0, 1, 2, \dots$, then $|\tau(\lambda x) - \tau(\lambda)| \leq n + 1 \leq \frac{|\ln x|}{\ln 2} + 1$. Recall now that $\tau(x) = o(\ln x)$, as $x \rightarrow \infty$. Then we can find constants M_0, M_1 such that $|\tau(x)| \leq M_0 |\ln x| + M_1$, $x > 0$. Therefore, if $x \in [\frac{1}{2^{n+1}}, \frac{1}{2^n}]$ but $\lambda < 2^n \lambda_0$, then $\lambda < \lambda_0/x$ and so $|\tau(\lambda x) - \tau(\lambda)| \leq 2(M_0 |\ln \lambda| + M_1) \leq 2(M_0 |\ln x| + M_0 |\ln \lambda_0| + M_1)$. Summarizing, if $A_0 = \max\{2M_0, 1/\ln 2\}$, $A_1 = \max\{2(M_0 |\ln \lambda_0| + M_1), 1\}$, then $|\tau(\lambda x) - \tau(\lambda)| \leq A_0 |\ln x| + A_1$ for $x > 0$ and $\lambda \geq \lambda_0$. $\square$

The asymptotically homogeneous functions of degree 0 do not have to be smooth, i.e., C^∞ . However, we have

Lemma 3. *Let τ be an asymptotically homogeneous function of degree 0, continuous in $[0, \infty)$. Then there exists a function σ , asymptotically homogeneous of degree 0, smooth in $[0, \infty)$, such that*

$$(2.5) \quad \tau(x) = \sigma(x) + o(1), \quad \text{as } x \rightarrow \infty.$$

Proof. Define $\sigma(\lambda) = \int_0^\infty \tau(\lambda x) \phi_0(x) dx$, $\lambda \geq 1$, where $\phi_0 \in \mathcal{S}$ satisfies $\int_0^\infty \phi_0(x) dx = 1$, and extend to $[0, +\infty)$ in any smooth way. Then (2.4) gives (2.5). $\square$

3. PRIMITIVES OF A DISTRIBUTIONALLY NULL SEQUENCE

If $\{f_n\}$ is a sequence of continuous functions that converge uniformly to zero on an interval of the form $[-A, A]$, then the sequence of primitives $\{\int_0^x f_n(t) dt\}$ also converges uniformly to zero while, in general, the sequence of derivatives $\{f'_n\}$, even if defined, does not. On the other hand, the situation with the distributional convergence is the opposite: if $f_n \rightarrow 0$ in $\mathcal{S}'$, then also $f'_n \rightarrow 0$ in $\mathcal{S}'$ but the sequence of primitives $\{\int_0^x f_n(t) dt\}$, even if defined, might be divergent. We now turn our attention to the study of this last problem.

Let $F \in \mathcal{S}'$. If F is integrable near $x = 0$, then we can define the primitive $\int_0^x F(t) dt$. If F is more singular near $x = 0$, then there is no canonical way to

define $\int_0^x F(t) dt$, that is, there is no canonical way to single out a primitive of F that vanishes at $x = 0$. However, if F is locally even near $x = 0$, it has a unique locally odd primitive, which we denote as $\int_0^x F(t) dt$. Observe that with this notation $\int_0^x \delta(t) dt = \frac{1}{2} \operatorname{sgn} x$, where $\operatorname{sgn} x = x/|x|$, $x \neq 0$, is the sign function. We shall use the notation $\int_0^x F(t) dt$ so long as $F = F_0 + F_1$, where F_0 is locally integrable near $x = 0$ and where F_1 is locally even.

Let $\{F_n\}_{n=1}^\infty$ be a sequence of distributions of $\mathcal{S}'$. Suppose that $\lim_{n \rightarrow \infty} F_n = 0$ in $\mathcal{S}'$. Then [10] there are primitives of the F_n that also tend to zero. In particular, if $\int_0^x F_n(t) dt$ is defined for each n , there are constants c_n such that $\int_0^x F_n(t) dt = c_n + o(1)$, as $n \rightarrow \infty$ in $\mathcal{S}'$. The example $F_n(x) = -2nxe^{-x^2}(1 - e^{-x^2})^{n-1}$, $\int_0^x F_n(t) dt = 1 - (1 - e^{-x^2})^n = 1 + o(1)$ shows that the c_n 's cannot be replaced by 0, in general. Observe also that if the F_λ depend smoothly on the parameter λ and $F_\lambda = o(1)$, as $\lambda \rightarrow \infty$, then $\int_0^x F_\lambda(t) dt = \sigma(\lambda) + o(1)$, where σ is smooth.

Lemma 4. *Let $F_0 \in \mathcal{S}'$ be a Radon measure such that $\int_0^x F_0(t) dt$ is defined. Define the distributions F_n , $n \geq 1$, recursively by $F_n(x) = \int_0^x F_{n-1}(t) dt$, $n \geq 1$. Suppose*

$$(3.1) \quad F_0(\lambda x) = o(1/\lambda), \quad \text{as } \lambda \rightarrow +\infty, \text{ in } \mathcal{S}'.$$

Then there exists an asymptotically homogeneous function of degree 0, $\sigma(\lambda)$, such that

$$(3.2) \quad F_n(\lambda x) = \frac{\lambda^{n-1} x^{n-1} \sigma(\lambda)}{(n-1)!} + o(\lambda^{n-1}), \quad \text{as } \lambda \rightarrow +\infty, \text{ in } \mathcal{S}',$$

for $n \geq 1$. There exists n_0 such that the convergence in (3.2) is uniform on compacts for $n \geq n_0$. Conversely, if (3.2) holds for some $n \geq 1$, then $F_0(\lambda x) = o(1/\lambda)$ in $\mathcal{S}'$.

Proof. Suppose $F_0(\lambda x) = o(1/\lambda)$ in $\mathcal{S}'$. Then there exists a smooth function $\sigma(\lambda)$ such that $F_1(\lambda x) = \sigma(\lambda) + o(1)$, as $\lambda \rightarrow \infty$, in $\mathcal{S}'$. Replacing λx by $\lambda x a$ and grouping in two different ways, we obtain $\sigma(a\lambda) = \sigma(\lambda) + o(1)$, as $\lambda \rightarrow \infty$ for each $a > 0$. Thus σ is asymptotically homogeneous of degree 0. Hence (3.2) holds for $n = 1$. Suppose now it holds for some $n \geq 1$. Then integrating again we obtain $F_{n+1}(\lambda x) = \lambda^n x^n \sigma(\lambda)/n! + \rho(\lambda) + o(\lambda^n)$, as $\lambda \rightarrow \infty$, for some function ρ . Evaluating at $\lambda a x$ thus yields $\rho(\lambda a) = \rho(\lambda) + o(\lambda^n)$ and thus by Lemma 1, applied to $\lambda^{-n} \rho(\lambda)$, it follows that $\rho(\lambda) = o(\lambda^n)$ and (3.2) is obtained for $n + 1$.

That the convergence in (3.2) is uniform for x in compacts if n is large follows by the definition of the convergence of distributions.

The converse is obtained by differentiating (3.2) n times with respect to x . $\square$

Observe that if F_0 is even, then F_1 is odd and thus $\sigma(\lambda) = o(1)$, as $\lambda \rightarrow \infty$, and (3.2) becomes $F_n(\lambda x) = o(\lambda^{n-1})$, as $\lambda \rightarrow \infty$. The same conclusion is obtained if $\operatorname{supp} F_0 \subseteq [0, \infty)$.

The sequence of functions $\{F_n\}_{n=0}^\infty$ is also related to Cesàro summability. The following lemma is an immediate corollary of the results proved in [6, p. 110].

Lemma 5. *Let F_0 be a function defined for $x > 0$ that, suitably extended to $\mathbb{R}$, defines an element of $\mathcal{S}'$ for which $\int_0^x F_0(t) dt$ is defined. Then $\lim_{x \rightarrow +\infty} F_0(x) = \gamma$ (C, n) if and only if $\lim_{x \rightarrow \infty} n! x^{-n} F_n(x) = \gamma$. $\square$*

4. THE MAIN RESULT

In this section we apply the results of the previous sections to characterize the Fourier series of the periodic distributions having a distributional point value.

Lemma 6. Let $f \in \mathcal{S}'$ be a periodic distribution of period 2π and let $\sum_{n=-\infty}^{\infty} a_n e^{in\theta}$ be its Fourier series. Let $\theta_0 \in \mathbb{R}$. Then

$$(4.1) \quad f(\theta_0) = \gamma, \quad \text{in } \mathcal{D}',$$

if and only if

$$(4.2) \quad \sum_{n=-\infty}^{\infty} a_n e^{in\theta_0} \delta(\lambda x - n) = \frac{\gamma \delta(x)}{\lambda} + o(1/\lambda), \quad \text{as } \lambda \rightarrow \infty, \text{ in } \mathcal{S}'.$$

Proof.

$$\begin{aligned} f(\theta_0) &= \gamma, \quad \text{in } \mathcal{D}' \\ &\Leftrightarrow \lim_{\varepsilon \rightarrow 0} \langle f(\theta_0 + \varepsilon\theta), \phi(\theta) \rangle = \langle \gamma, \phi(\theta) \rangle \quad \forall \phi \in \mathcal{D} \\ &\Leftrightarrow \lim_{\varepsilon \rightarrow 0} \langle f(\theta_0 + \varepsilon\theta), \phi(\theta) \rangle = \langle \gamma, \phi(\theta) \rangle \quad \forall \phi \in \mathcal{S} \\ &\Leftrightarrow \lim_{\varepsilon \rightarrow 0} \sum_{n=-\infty}^{\infty} a_n e^{in\theta_0} \langle e^{in\varepsilon\theta}, \phi(\theta) \rangle = \gamma \int_{-\infty}^{\infty} \phi(x) dx \quad \forall \phi \in \mathcal{S} \\ &\Leftrightarrow \lim_{\varepsilon \rightarrow 0} \sum_{n=-\infty}^{\infty} a_n e^{in\theta_0} \hat{\phi}(n\varepsilon) = \gamma \hat{\phi}(0) \quad \forall \phi \in \mathcal{S} \\ &\Leftrightarrow \lim_{\varepsilon \rightarrow 0} \sum_{n=-\infty}^{\infty} a_n e^{in\theta_0} \psi(n\varepsilon) = \gamma \psi(0) \quad \forall \psi \in \mathcal{S} \\ &\Leftrightarrow \lim_{\varepsilon \rightarrow 0} \sum_{n=-\infty}^{\infty} a_n e^{in\theta_0} \delta(x - n\varepsilon) = \gamma \delta(x) \quad \text{in } \mathcal{S}' \\ &\Leftrightarrow \sum_{n=-\infty}^{\infty} a_n e^{in\theta_0} \delta(\lambda x - n) = \frac{\gamma \delta(x)}{\lambda} + o(1/\lambda), \quad \text{as } \lambda \rightarrow \infty \text{ in } \mathcal{S}'. \quad \square \end{aligned}$$

We are now ready to give our main result.

Theorem. Let $f \in \mathcal{S}'$ be a periodic distribution of period 2π and let $\sum_{n=-\infty}^{\infty} a_n e^{in\theta}$ be its Fourier series. Let $\theta_0 \in \mathbb{R}$. Then

$$(4.3) \quad f(\theta_0) = \gamma, \quad \text{in } \mathcal{D}',$$

if and only if there exists k such that

$$(4.4) \quad \lim_{x \rightarrow \infty} \sum_{-x \leq n \leq ax} a_n e^{in\theta_0} = \gamma \quad (C, k)$$

for each $a > 0$.

Proof. Let $F_0(x) = \sum_{n=-\infty}^{\infty} a_n e^{in\theta_0} \delta(x - n)$ and define $F_1, F_2, \dots$ recursively by $F_{n+1}(x) = \int_0^x F_n(t) dt$. Let $a > 0$, put $G_0(x) = G_0(a, x) = \sum_{-x \leq n \leq ax} a_n e^{in\theta_0}$ and define $G_1, G_2, \dots$ recursively by $G_{n+1}(x) = \int_0^x G_n(t) dt$. Observe that (4.4) means that $\lim_{x \rightarrow \infty} k! x^{-k} G_k(x) = \gamma$.

Suppose first that $f(\theta_0) = \gamma$. Then $F_0(\lambda x) = \gamma \delta(\lambda x) + o(1/\lambda)$, as $\lambda \rightarrow \infty$, in $\mathcal{S}'$. Using Lemma 4, there exists an asymptotically homogeneous function of degree 0 such that if $n \geq 1$, then

$$F_n(\lambda x) = \frac{\gamma \operatorname{sgn}(\lambda x) (\lambda x)^{n-1}}{2(n-1)!} + \frac{\sigma(\lambda) (\lambda x)^{n-1}}{(n-1)!} + o(\lambda^{n-1}), \quad \text{as } \lambda \rightarrow \infty, \text{ in } \mathcal{S}'$$

and there exists n_0 such that if $n \geq n_0$ this holds uniformly on $x \in [-A, A]$ for $A > 0$. Thus, if $n \geq n_0$,

$$\begin{aligned} G_{n-1}(x) &= a^{1-n} F_n(ax) - (-1)^{1-n} F_n(-x) \\ &= \frac{\gamma x^{n-1}}{2(n-1)!} + \frac{\sigma(x)}{(n-1)!} + \frac{\gamma}{2(n-1)!} - \frac{\sigma(x)}{(n-1)!} + o(x^{n-1}) \\ &= \frac{\gamma x^{n-1}}{(n-1)!} + o(x^{n-1}) \end{aligned}$$

and (4.4) follows with $k = n - 1$.

Conversely, suppose (4.4) holds. Let $n = k + 1$, so that

$$\lim_{x \rightarrow \infty} (n-1)! x^{1-n} G_{n-1}(x) = \gamma.$$

Define $\tau(x) = (n-1)! x^{1-n} F_n(x) - \gamma/2$. Then τ is asymptotically homogeneous of degree 0. Thus there exists a smooth asymptotically homogeneous function of degree 0 such that $\tau(x) = \sigma(x) + o(1)$, as $x \rightarrow \infty$. It follows that

$$F_n(x) = \frac{\gamma \operatorname{sgn} x x^{n-1}}{2(n-1)!} + \frac{\sigma(x) x^{n-1}}{(n-1)!} + o(x^{n-1}), \quad \text{as } |x| \rightarrow \infty,$$

and therefore, by Lemma 2,

$$F_n(\lambda x) = \frac{\gamma \operatorname{sgn}(\lambda x)(\lambda x)^{n-1}}{2(n-1)!} + \frac{\sigma(\lambda)(\lambda x)^{n-1}}{(n-1)!} + o(\lambda^{n-1}), \quad \text{as } \lambda \rightarrow \infty, \text{ in } \mathcal{S}'.$$

Differentiating n times we obtain $F_0(\lambda x) = \gamma \delta(\lambda x) + o(1/\lambda)$, as $\lambda \rightarrow \infty$, as required. $\square$

It is interesting to observe that (4.4) holds uniformly on $a \in [A, B]$ if $[A, B] \subseteq (0, \infty)$ but that it is not necessary to assume such uniform behavior to obtain (4.3).

Notice also that the theorem remains valid if (4.4) holds uniformly for $a \in H$ where H is a set dense in some interval, $\overline{H} = [A, B]$, $0 < A < B < \infty$. In particular, $f(\theta_0) = \gamma$ if and only if

$$(4.5) \quad \lim_{N \rightarrow \infty} \sum_{n=-\lfloor N/p \rfloor}^{\lfloor N/q \rfloor} a_n e^{in\theta_0} = \gamma \quad (C, k)$$

for each $p, q \in \{1, 2, 3, \dots\}$, uniformly if $A < p/q < B$ for $[A, B] \subseteq (0, \infty)$. Observe that (4.5) considers the convergence, in the Cesàro sense, of sequences.

The following consequences are worth recording.

Corollary 1. *If f is symmetric about $\theta = \theta_0$, i.e., $f(\theta - \theta_0) = f(\theta_0 - \theta)$, then $f(\theta_0) = \gamma$ in $\mathcal{D}'$ if and only if $\sum_{n=-\infty}^{\infty} a_n e^{in\theta_0} = \gamma$ (C).* $\square$

Corollary 2. *If f is antisymmetric about $\theta = \theta_0$, i.e., $f(\theta - \theta_0) = -f(\theta_0 - \theta)$, then $f(\theta_0) = \gamma$ in $\mathcal{D}'$ if and only if there is a function σ , asymptotically homogeneous of degree 0, such that $\sum_{n=1}^N a_n e^{in\theta_0} = \sigma(N) + o(1)$ (C), as $N \rightarrow \infty$. In this case $\gamma = 0$.* $\square$

Corollary 3. *If the Fourier series of f is of the power series type, i.e., $f(\theta) = \sum_{n=0}^{\infty} a_n e^{in\theta}$, then $f(\theta_0) = \gamma$ in $\mathcal{D}'$ if and only if $\sum_{n=0}^{\infty} a_n e^{in\theta_0} = \gamma$ (C).* $\square$

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