PROCEEDINGS OF THE
AMERICAN MATHEMATICAL SOCIETY
Volume 124, Number 4, April 1996

SIMPLE PROOF OF CALABI-BERNSTEIN'S THEOREM ON MAXIMAL SURFACES

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(Communicated by Christopher Croke)

Dedicated to the memory of Professor P. Bobillo

The following uniqueness result, called Calabi-Bernstein's Theorem on maximal surfaces, is well known ([Ca], [C-Y], [K], [E-R]).

Theorem. *The only entire solutions to the maximal surface equation*

$$\operatorname{div}\left(\frac{Du}{\sqrt{1-|Du|^2}}\right)=0, \qquad |Du|<1,$$

are affine functions.

In the references cited above, this result appears as either a particular case of some much more general theorems or stated in terms of local complex representation of the surface. However, a direct simple proof would be desirable to be easily understood for beginning researchers. The proof we present here uses only Liouville's Theorem on harmonic functions on $\mathbb{R}^2$. Thus, it is simple and complex function theory is not needed. This proof is inspired by [Ch]. Roughly, the key steps of our proof are: (1) On any maximal surface there exists a positive harmonic function, which is constant if and only if the surface is totally geodesic. (2) The metric of any spacelike graph is globally conformally related to a metric g^*, which is complete when the graph is entire. (3) On any maximal graph the metric g^* is flat.

1. Preliminaries

Consider the Lorentz-Minkowski space $\mathbb{L}^3$ with its Lorentzian metric $\langle\ ,\ \rangle = dx_1^2+dx_2^2-dx_3^2$, given in the usual coordinate system. Let $x : M \to \mathbb{L}^3$ be a spacelike immersion of a two-dimensional manifold M in $\mathbb{L}^3$. Note that M must be orientable. Let N be a globally defined unit timelike normal vector field on M. Suppose that the mean curvature of x vanishes; then M is called a maximal surface in $\mathbb{L}^3$.

For each vector $a \in \mathbb{L}^3$ we can consider on M the smooth function $\langle N, a\rangle$. Let a^T be the vector field on M induced from the tangential component of a at any

Received by the editors August 11, 1994.
1991 *Mathematics Subject Classification.* Primary 53C42, 53C50.
Key words and phrases. Maximal surface, entire maximal graph, Lorentz-Minkowski space.
Research partially supported by DGICYT Grant No. PB91-0731.

point of M, that is, $a^T = a + \langle N, a\rangle N$. Standard computations give the identities

$$\Delta\langle N, a\rangle = \mathrm{tr}(A^2)\langle N, a\rangle, \tag{1}$$

$$\nabla\langle N, a\rangle = -A(a^T), \tag{2}$$

where A is the Weingarten endomorphism associated to N, and Δ and ∇ are respectively the Laplacian and the gradient relative to the induced Riemannian metric g on M.

From (2) we obtain

$$|\nabla\langle N, a\rangle|^2 = (1/2)\,\mathrm{tr}(A^2)\{\langle N, a\rangle^2 + \langle a, a\rangle\}. \tag{3}$$

Now, choose $a \neq 0$ satisfying $\langle a, a\rangle = 0$. Note that in this case $\langle N, a\rangle$ never vanishes and thus we may assume $\langle N, a\rangle > 0$. From (1) and (3) we get

$$\Delta(1/\langle N, a\rangle) = 0. \tag{4}$$

On the other hand, if we choose now $b \in \mathbb{L}^3$ such that $\langle b, b\rangle = -1$ and $\langle N, b\rangle > 0$, then, using again (1) and (3) we obtain

$$\Delta \log(\langle N, b\rangle + 1) = (1/2)\,\mathrm{tr}(A^2) \tag{5}$$

which means, taking into account the Gauss equation $2K = \mathrm{tr}(A^2)$ for x, that the metric $g^* = (\langle N, b\rangle + 1)^2 g$ on M is flat.

2. Proof of the Theorem

We begin by observing that on a spacelike graph in $\mathbb{L}^3$ of any entire function $u = u(x, y), |Du| < 1$, the metric $g' = (1/(1 - |Du|^2))g$ is complete. This easily follows from the inequality $L' \geq (1/\sqrt{2})L_0$, where L' (resp. L_0) denotes the length of a curve on the graph with respect to g' (resp. the metric induced from the usual Riemannian one of $\mathbb{R}^3$). Now take $b = (0, 0, -1)$ so that $\langle N, b\rangle = 1/\sqrt{1 - |Du|^2}$. Consider now the Riemannian metric g^* introduced above. This metric is complete on a spacelike graph of any entire function. Assume the graph is maximal. On the other hand, from (5) we see that g^* is flat. Using Cartan's Theorem we have a global isometry from the Euclidean plane $\mathbb{R}^2$ onto the maximal entire graph endowed with the metric g^*. The invariance of harmonic functions under conformal changes of metric and this isometry permit us to induce $1/\langle N, a\rangle$ on a positive harmonic function on $\mathbb{R}^2$ which must be constant by Liouville's Theorem. Hence $\langle N, a\rangle$ is constant on the graph, which implies, using (1), that the maximal graph is totally geodesic.

3. Remark

Consider as in §1, a maximal surface M in $\mathbb{L}^3$. The metric g^* on M is also complete if the induced metric g is complete. Thus, the same reasoning as in the previous section gives a proof of the well-known parametric version of this Theorem: "The only complete maximal surfaces in $\mathbb{L}^3$ are the spacelike planes."

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