NON-SIMPLE LINKS WITH TUNNEL NUMBER ONE

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ABSTRACT. We determine all non-simple links which admit an unknotting tunnel, i.e. links which contain an essential annulus or torus in its exterior and have tunnel number one.

A link l is *non-simple* if there is an essential annulus or torus properly embedded in the exterior of l , $E(l)$, that is, there is an annulus or torus which is incompressible, ∂ -incompressible, and non ∂ -parallel. A link l admits an unknotting tunnel if there is a properly embedded arc τ in $E(l)$ such that $E(l \cup \tau)$ is a handlebody; in that case we say that l has tunnel number one. If l is not a knot and admits an unknotting tunnel, then necessarily l has two components, say $l = l_1 \cup l_2$, and any unknotting tunnel for l connects l_1 to l_2 .

In this paper we give a complete list of all non-simple links which admit an unknotting tunnel. Knots admitting unknotting tunnels are classified in [MS], [E]. First we show that if l is a non-simple link which has an unknotting tunnel τ , then there is an essential annulus A in $E(l)$ disjoint from τ . This divides the non-simple links into two types, depending if $\partial A = a_1 \cup a_2$ lie on different, or on the same component of $\partial E(l)$. In Theorem 1 (2) we give a list of the non-simple links having an annulus as in the first (second) case.

Lemma. *Let l be a non-simple link with an unknotting tunnel τ . Then there exist an essential annulus A in $E(l)$ such that $A \cap \tau = \emptyset$.*

Proof. Let $M = E(l \cup \tau)$. M is a genus 2-handlebody. $E(l)$ is obtained by adding a 2-handle to M along a curve α on ∂M which is a cocore of the tunnel τ . Clearly ∂M is compressible, $\partial M \setminus \alpha$ is incompressible, and α is a separating curve on ∂M . If there is an essential annulus or torus in $E(l)$, then by Theorems 1 and 2 of [E], there is an essential (incompressible, not parallel to a surface in ∂M) annulus A in M disjoint from α , that is, an essential annulus in $E(l)$ disjoint from τ . $\square$

Theorem 1. *Let l be a non-simple link with unknotting tunnel τ , and A an essential annulus disjoint from τ . Suppose that $\partial A = a_1 \cup a_2$ lie on different components of $\partial E(l)$. Then one of the following cases occurs:*

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FIGURE 1

FIGURE 2

- (i) If a_1 is a meridian of $N(l_1)$ and a_2 is (necessarily) isotopic to l_2 in $N(l_2)$, then l is a connected sum of a two-bridge knot k and the Hopf link.
- (ii) If a_1 is isotopic to l_1^n ($n \geq 2$) in $N(l_1)$ and a_2 is isotopic to l_2 in $N(l_2)$, then l_2 is a torus knot and l_1 is its core.
- (iii) If a_i is isotopic to l_i in $N(l_i)$ ($i = 1, 2$), then l is a torus link with type $(2, 2p)$, $p \in \mathbb{N}$.
- (iv) If a_i is isotopic to $l_i^{n_i}$ ($n_i \geq 2$) in $N(l_i)$ ($i = 1, 2$), then l is the Hopf link.

Proof. By the definition of a tunnel, $E(l \cup \tau)$ is a genus two handlebody V . Since each component of ∂A lies on a different component of $\partial E(l)$ and A is essential in $E(l)$, A is a non-separating essential annulus in V . According to [K, Lemma 3.2 (ii)], we can assume that the pair (V, A) is like in Figure 1.

Note that there exists a meridian disk Δ of V such that $\Delta \cap A = \emptyset$. Now, we consider the knot k in V indicated in Figure 1. Then $V' = V \setminus N^\circ(k)$ is as shown in Figure 2.

Let α be a simple closed curve on ∂V which is a cocore of the tunnel τ . Since each component of ∂A lie on a different component of $E(l)$, α separates a_1 from a_2 on ∂V . Let B_1, B_2 be the annuli indicated in Figure 2. Then $\{B_1, B_2\}$ cuts V' into a genus two handlebody V'' . Let a'_i, a''_i be copies of a_i ($i = 1, 2$) on $\partial V''$. (See Figure 3.)

We can consider $(V'', \partial N(k))$ as a trivial tangle. Since the curve α separates $\{a'_1, a''_1\}$ from $\{a'_2, a''_2\}$, it follows that α is as shown in Figure 4(i). Then, we can assume that V' is as shown in Figure 4(ii). By attaching a 2-handle to V' along α , we have $\Sigma = E(l) \setminus N^\circ(k)$. Now, after moving $\partial N(l_2)$ through k along the annulus

FIGURE 3

FIGURE 4

FIGURE 5

A , Σ looks like in Figure 5. $\partial N(k)$ as in Figure 5 determines the exterior of a knot K in S^3 . Note that K is a two-bridge knot.

Case (i). Since a_2 is isotopic to l_2 in $N(l_2)$, $\Sigma' = \Sigma \cup N(l_2)$ is indicated in Figure 6(i).

We can exchange $\partial N(l_1)$ for $\partial N(k)$ by a homeomorphism of Σ . (See Figure 6(ii). If we cut Σ' along the annulus C shown in Figure 6, it is easy to find the homeomorphism. It can also be seen applying the bulb trick, where $\partial N(l_1)$ becomes knotted while $\partial N(k)$ is being unknotted.) But because a_1 is a meridian of $N(l_1)$, it follows that (S^3, l) is as shown in Figure 6(iii), by a surgery description. The manifold is S^3 , so $s/r = \pm 1/n$, $n \in \mathbb{N}$.

FIGURE 6

FIGURE 7

This completes the proof of Case (i).

Case (ii). Since a_2 is isotopic to l_2 in $N(l_2)$, $\Sigma' = \Sigma \cup N(l_2)$ is indicated in Figure 6(i). Let $X = N(l_1) \cup N(C) \cup N(k)$, $c_1 = N(l_1) \cap C$, $c_2 = N(k) \cap C$. Then $S^3 = X \cup E(X)$, $E(X)$ is homeomorphic to $E(K)$, and a_1 is a meridian of $E(K)$. Now, we consider the following two cases:

- (a) K is a non-trivial two-bridge knot, or
- (b) K is a trivial knot.

Case (ii-a). X must be a solid torus, for otherwise $X \cup E(X)$ could not be S^3 . Since $a_1 \cap C = \emptyset$, a_1 is a curve on ∂X . Then a_1 is not a meridian of X , because a_1 is isotopic to l_1^n ($n \geq 2$) in $N(l_1)$. Now $S^3 = X \cup_{a_1} E(X)$. This is a contradiction, because every non-trivial two-bridge knot has Property P by [T]. Hence Case (ii-a) does not occur.

Case (ii-b). We can consider X as a union of two solid tori attached along an annulus. So X must be a torus knot exterior, and l_1 is its core. Since K is trivial, $E(X)$ ($\cong E(K)$) is a solid torus and l_2 is its core. Then, we have that l_2 is a torus knot and l_1 is its core. This completes the proof of Case (ii).

Case (iii). Since a_i is isotopic to l_i in (N_i) ($i = 1, 2$), $(S^3, l_1 \cup l_2)$ is indicated in Figure 7 by a surgery description, where s/r is a surgery coefficient.

Since A is incompressible, this Dehn surgery on K must be non-trivial. Every non-trivial two-bridge knot has Property P by [T], so K must be trivial, and r/s must be $\pm 1/n$. Then l is a torus link, with type $(2, 2n)$, $n \in \mathbb{N}$.

This completes the proof of Case (iii).

Case (iv). Let $Y = N(l_1) \cup N(A) \cup N(l_2)$. Then $S^3 = Y \cup E(Y)$. Since a_i is isotopic to l_1^n ($n \geq 2$), Y must be a non-trivial torus knot exterior and $E(Y)$ must

be a solid torus. Since l_1 (resp. l_2) is the core of $N(l_1)$ (resp. $N(l_2)$), l is the Hopf link.

This completes the proof of Case (iv). $\square$

Note that any link satisfying one of Cases (i)–(iv) of Theorem 1 is non-simple and has an unknotting tunnel.

We describe two classes of non-simple links which admit unknotting tunnels.

(1) Let R be a standard solid torus in S^3 . Let $\mathcal{C}$ be the collection of all annuli in ∂R , which winds at least twice algebraically in the complementary solid torus of R in S^3 , so for any such annulus S , a component of ∂S is a curve on ∂R of slope (p, q) , for some relative prime integers p and q with $|p| \geq 2$. Let l_1 be the core of R . Let k be a nontrivial knot in $S^3 \setminus l_1$ which has a 1-bridge presentation with respect to some annulus S of the collection $\mathcal{C}$ (cf. [E, 4.3]), that is: (a) k can be isotoped to intersect transversally ∂R in two points, both points lying in S . Then k is divided into two arcs k_1, k_2 . (b) k_i , $i = 1, 2$, can be projected to S to a single arc, that is, there is a disk D_i such that $\partial D_i = k_i \cup \gamma_i$, where γ_i is a single arc in S , and $\text{int} D_i \cap \partial R = \emptyset$, $\text{int} D_i \cap k = \emptyset$. Let $l = l_1 \cup l_2$ be a link such that l_1 is as above and $l_2 = k$. We say that l belongs to the class $\mathcal{A}$.

If l is in the class $\mathcal{A}$, then l_2 has a 1-bridge presentation with respect to some annulus $S \subset \partial R$, and it follows that l_2 can be isotoped to lie in $N(S)$. Let $T = \partial N(S)$. If l_2 is not parallel to the core of S , then the torus T is essential in $E(l)$. Note that in any case there is an essential annulus A in $E(l)$, which has its boundary on $\partial E(l_1)$; this is as follows: Let $\partial S = s_1 \cup s_2$. There are annuli A_1 and A_2 in R , joining the curves s_1 and s_2 to respective curves on $\partial N(l_1)$; let $A = (T \setminus R) \cup A_1 \cup A_2$. The link l admits an unknotting tunnel, which is a straight arc joining l_1 and l_2 , which is disjoint from the annulus A .

Note that if $l \in \mathcal{A}$ and $q \geq 2$, then l_2 is a satellite knot with tunnel number one.

The following is an alternative description of the class of links $\mathcal{A}$; it follows from [E, 4.8] that these descriptions are equivalent: Let k_0 be a torus knot of type (p, q) in S^3 , where $|p| \geq 2$, so k_0 may be trivial. Let $L = k_1 \cup k_2$ be a 2-bridge link of type (α, β) in S^3 . Since k_2 is a trivial knot, there is an orientation-preserving homeomorphism

$$f : S^3 \setminus N^\circ(k_2) \rightarrow N(k_0)$$

which takes a meridian $m_2 \subset \partial N(k_2)$ of k_2 to a fiber

$$h \subset \partial N(k_0) = \partial(S^3 \setminus N^\circ(k_0))$$

of the Seifert fibration $D(-r/p, s/q)$ of S^3 . We consider the link $l = l_1 \cup l_2$, where l_1 is the core of the torus knot k_0 , and l_2 is the knot $f(k_1)$. The link l is determined by the symbol $(\alpha, \beta; p, q)$. In Figure 8 a member of this family is shown.

(2) Let k be a non-trivial (p, q) -torus knot or a (p, q) -cable of a (p_0, q_0) -torus knot, where $p = qp_0q_0 \pm 1$; i.e. k is a torus knot or a cable knot with tunnel number one (cf. [MS], [Mo], [E, 4.6]).

Let k be a torus knot; then any unknotting tunnel for k is isotopic to one of the tunnels τ_u, τ_v, τ_0 described in [BRZ] or [MS]. There is an essential annulus A in $E(k)$, which divides it into two solid tori Q_1, Q_2 . Choose representatives of τ_u, τ_v , and τ_0 , which are arcs properly embedded in $E(k)$, are disjoint from A , τ_u (τ_v) is contained in Q_1 (Q_2), and τ_0 is contained in either of Q_1 or Q_2 , for it is isotopic to an arc lying on A .

FIGURE 8

Let k be a cable of a torus knot as above. Any unknotting tunnel for k is isotopic to one of two unknotting tunnels described in [MS], which we denote by τ_1, τ_2 . There is an essential annulus A in $E(k)$, which divide it into a solid torus Q_1 , and the exterior $E(k_0)$ of a torus knot k_0 . Choose representatives of τ_1 and τ_2 , which are arcs properly embedded in $E(k)$, are disjoint from A , and are (necessarily) contained in $E(k_0)$.

Let k and A be as above. ∂A divides $\partial E(k)$ into two annuli; let B be one of these annuli (either one if k is a torus knot, and the one contained in $E(k_0)$ if k is a cable of a torus knot k_0). Embed $B \times I$ into $E(k)$, where $I = [0, 1]$ (and such that $A \cap (B \times I) = \partial B \times \{0\}$). Clearly

$$\text{cl}(E(k) \setminus (B \times I)) = E(k).$$

If τ is one of the chosen tunnels of k , assume that its endpoints lie on $B \times \{1\}$, say these are $(x_0, 1)$ and $(x_1, 1)$, where x_0 and x_1 are distinct points in the interior of B . Let k' be a knot in $S^3 - k$ divided into 2 simple arcs, k_1 and k_2 , such that k_1 is an arc lying on $B \times \{1\}$ with endpoints in x_0 and x_1 , and k_2 is one of the standard tunnels for k described above. Define a link of two components $l = l_1 \cup l_2$, where $l_1 = k$ and $l_2 = k' = k_1 \cup k_2$. l is a non-simple link, for A is an essential annulus in $E(l)$; l has an unknotting tunnel, which is the arc $\{x_0\} \times I$. We say that l belongs to the class $\mathcal{B}$.

Note that arcs on an annulus with ends on two fixed points can be parameterized by the rational fractions α/β , where α is odd and β is even. (Pair of arcs in a sphere with endpoints in four fixed points are parameterized by $Q \cup \{1/0\}$; such arcs are divided in three classes, depending of the points they join, and each class is parameterized by a class of fractions α/β , where this class is determined by the parity of α and β . Arcs in an annulus with endpoints in two fixed points are in one-to-one correspondence with any of the classes of arcs of the sphere. We can choose then the class parameterized by fractions α/β where α is odd and β is even.) A link $l = l_1 \cup l_2$ in the class $\mathcal{B}$ can be described by a symbol $(p, q; \alpha, \beta; \tau_i)$ or a symbol $(p, q; p_0, q_0; \alpha, \beta; \tau_i)$, depending if l_1 is a (p, q) -torus or a (p, q) -cable of a (p_0, q_0) -torus knot. (α, β) and τ_i denote the types of arcs in which l_2 is divided, where τ_i is one of the standard tunnels for l_1 . In Figure 9 a member of $\mathcal{B}$ is shown.

Theorem 2. *Let l be a non-simple link with unknotting tunnel τ , and A an essential annulus disjoint from τ . Suppose that $\partial A = a_1 \cup a_2$ lie on the same component*

FIGURE 9

of $\partial E(l)$, say it lies on $\partial N(l_1)$. Then one of the following cases occurs:

- (i) If A is inessential in $E(l_1)$, then l is in $\mathcal{A}$.
- (ii) If A is essential in $E(l_1)$, then l is in $\mathcal{B}$.

Proof. Suppose that $\partial A \subset \partial N(l_1)$. A is incompressible in $M = E(l \cup \tau)$, and it is separating. $M \setminus N^\circ(A)$ consists of two handlebodies. a_1 cannot be a meridian of l_1 , for in this case l would be a non-trivial connected sum $l = k_1 \# k_2$, where say, k_1 is a knot. But $E(k_1)$ is homeomorphic to a component of $M \setminus N^\circ(A)$, i.e., it is a handlebody; then k_1 is the trivial knot, which is a contradiction. Therefore a_1 is a curve of type (p, q) on $\partial N(l_1)$, $(p, q) \neq (1, 0)$.

There are two possibilities:

Case (i). A is inessential in $E(l_1)$. Then A is parallel in $E(l_1)$ to an annulus A' of type (p, q) in $\partial N(l_1)$. $E(l_1) \setminus N^\circ(A)$ has two components; one is a solid torus in which l_2 and τ lie, and the other is homeomorphic to $E(l_1)$, but this has to be a handlebody, so $E(l_1)$ is a solid torus and l_1 is a trivial knot. Therefore $|p| \geq 2$.

Note that $l_1 \cup \tau$ is an unknotting tunnel for l_2 , so by [E, 4.4], l_2 has a 1-bridge presentation with respect to $\partial N(l_1)$, and by the argument of [E, 4.7], l_2 has a 1-bridge presentation w.r.t. A' . This shows that $l \in \mathcal{A}$.

Case (ii). A is essential in $E(l_1)$. Therefore l_1 is a cable knot with tunnel number one, so l_1 is a torus knot or a (p, q) -cable of a (p_0, q_0) -torus knot, where $p = qp_0q_0 \pm 1$. A is the standard annulus properly embedded in $E(l_1)$. A divides $E(l_1)$ in two pieces; one is a solid torus, and the other is a manifold N homeomorphic to $E(k)$, where k is the companion of l_1 , so k is the trivial knot or a certain torus knot. We can assume that l_2 and τ lie on N . A divides $\partial E(l_1)$ on two annuli, let B be one of them.

Slide l_2 over τ to get an arc τ' properly embedded in $E(l_1)$. τ' is isotopic to one of the standard tunnels γ for $E(l_1)$.

Claim. There is an isotopy carrying τ' to γ which keeps the annulus A fixed.

The isotopy which carries τ' to γ transforms the annulus A into an annulus A' , which may intersect A . By looking at the intersections between the annuli and doing an innermost disk–outermost arc argument the isotopy can be deformed to one which fixes A , except possibly if $\gamma = \tau_0$ and τ' and γ lie on different components of $E(k) \setminus A$. But this can be fixed by assuming that if $\gamma = \tau_0$, then γ and τ' must

lie on the same component $E(k) \setminus A$ —there is no loss of generality in doing that. If $\gamma \neq \tau_0$, then necessarily γ and τ' lie on the same side of $E(k) \setminus A$.

Let x_0, x_1 be the endpoints of γ . We can assume that the endpoint of τ lying on l_1 coincides with one of the endpoints of γ , say this is x_0 , and that the isotopy between τ' and γ fixes that point. Consider an embedding of $B \times I$ into $E(k)$ so that $A \cap (B \times I) = \partial B \times \{0\}$. The isotopy carrying τ' to γ can be made in two steps. First, τ' is slid over B so that the complement of $B \times I$ is fixed, and keeping fixed the endpoint of τ' lying on x_0 . Second, τ' is isotoped to γ leaving fixed $B \times I'$, where $I' = [0, a]$ is a subinterval of I . $\gamma \cap (B \times I)$ consists of two straight arcs. This isotopy can be made in reverse order, in two steps. First, the endpoint of γ on x_0 is fixed, and the other is slid over B , keeping fixed the complement of $B \times I$. Second, γ is isotoped, keeping fixed $B \times I$, and finally τ' is slid over itself to get $l_2 \cup \tau$. This shows that l_2 can be decomposed into two arcs, k_1, k_2 , such that k_1 is an arc in $B \times I$, consisting of a monotonic arc going from $(x_1, 1)$ to $(x_0, 0)$ union τ , and k_2 is an arc in the complement of $B \times I$, which can be thought equal to γ . Note that the arc k_1 can be isotoped to a simple arc in $B \times \{1\}$, with endpoints $(x_0, 1)$ and $(x_1, 1)$. This shows that l belongs to the class $\mathcal{B}$. $\square$

Note. The list of links given in Theorem 1 is not disjoint from the list of links in Theorem 2. First note that case (iv) of Theorem 1 is contained in case (iii) of the same theorem. Links satisfying case (iii) of Theorem 1 also satisfy case (i) of Theorem 2. Links satisfying case (ii) of Theorem 1 satisfy both cases of Theorem 2. If l_1 is a (p, q) -cable of a (p_0, q_0) -torus knot, where $p = qp_0q_0 \pm 1$, and if l_2 is the core of the (p_0, q_0) -torus knot, then $l = l_1 \cup l_2$ satisfy both cases of Theorem 2. However these are all the intersections among the different cases.

Remark. Tunnel number one knots are prime [N], [S]; but it follows from Theorem 1(ii) that the same is not true for links. K. Morimoto [M] proved that the only composite links which admit unknotting tunnels are sums of 2-bridge knots with the Hopf link. In Theorem 1(i) we give a different proof of this fact. A third proof can be given using a stronger version of the lemma (which also follows from [E]) and an argument similar to the first part of the proof of Theorem 2. A. C. Jones [J] proved that if a composite link has a 2-generator fundamental group, then it has two summands, one is a knot and the other is the Hopf link. The fact that a rank 2 composite link has a Hopf link summand was previously known by F. González-Acuña and H. Short; see *Knot surgery and primeness*, Math. Proc. Cambridge Philos. Soc. **99** (1986), 89–102, page 96, line 32.

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