EIGENVALUE RATIOS FOR THE REGULAR STURM-LIOUVILLE SYSTEM

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ABSTRACT. Following the method of Ashbaugh-Benguria in Comm. Math. Phys. **124** (1989), 403–415; J. Differential Equations **103** (1993), 205–219, we prove an upper estimate of the arbitrary eigenvalue ratio (μ_m/μ_n) for the regular Sturm-Liouville system. This upper estimate is sharp for Neumann boundary conditions. We also discuss the sign of μ_1 and include an elementary proof of a useful trigonometric inequality first given in the aforementioned articles.

1. INTRODUCTION

A regular Sturm-Liouville system on $(0, 1)$ [4, p.300] is the regular Sturm-Liouville equation:

$$(1) \quad -(p(x)y')' + q(x)y = \mu w(x)y ,$$

where

$$(2) \quad p \in C^1[0, 1], \quad q, w \in C[0, 1], \quad p, w > 0;$$

with separated endpoint boundary conditions

$$(3) \quad \begin{cases} y(0) \cos \gamma - p(0)y'(0) \sin \gamma = 0, \\ y(1) \cos \delta - p(1)y'(1) \sin \delta = 0, \end{cases}$$

where $0 \leq \gamma < \pi$, $0 < \delta \leq \pi$. γ and δ are in fact the initial and final phases respectively in the classical Prüfer substitution.

It is well known that the above system has an increasing sequence of simple real eigenvalues

$$\mu_1 < \mu_2 < \cdots < \mu_n < \mu_{n+1} < \cdots ,$$

and the eigenfunction corresponding to μ_n has exactly n zeros in $(0, 1)$. Sometimes, an additional eigenvalue $\mu_0 < \mu_1$ may also occur, and the corresponding eigenfunction has no zero in $(0, 1)$.

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Recently, Ashbaugh and Benguria [3] proved an optimal upper estimate of the eigenvalue ratio λ_m/λ_n ($m > n \geq 1$) for the Dirichlet problem when $\gamma = 0$ and $\delta = \pi$, with the assumptions that

$$q \geq 0 \quad \text{and} \quad 0 < k \leq pw \leq K.$$

Based on a modified Prüfer substitution and trigonometric inequality, they proved that

$$\frac{\lambda_m}{\lambda_n} \leq \left(\left\lceil \frac{m}{n} \right\rceil\right)^2 \frac{K}{k},$$

with equality when and only when $q \equiv 0$ and $k = K$. This is a generalization of their previous work on one-dimensional Schrödinger operators [1, 2]. (In this paper, we adopt the following notation. Let $\lceil s \rceil$ denote the value of the ceiling function at s , i.e. $\lceil s \rceil = \min\{n \in \mathbf{N}, n \geq s\}$, and $\lfloor s \rfloor$ denote the value of the floor function at s , $\lfloor s \rfloor = \max\{n \in \mathbf{N}, n \leq s\}$.)

We extend their result to the general regular Sturm-Liouville system. The main idea is to allow the modified phase to take up a negative initial value and in the induction process we divide the interval $[0, 1]$ into two parts by the point where the phase first attains $\pi/2$. Our main theorem is

Theorem 1. *For the regular Sturm-Liouville system (1)–(3), if $q(x) \geq 0$ and $0 < k \leq pw \leq K$, then for any integers $m > n \geq 1$ discussed below, the eigenvalues μ_m and μ_n are positive, and*

$$\frac{\mu_m}{\mu_n} \leq l^2 \frac{K}{k}$$

where l is determined by the boundary conditions as follows:

$\pi \geq \delta \geq \pi/2$	$\pi \geq \pi - \gamma \geq \pi/2$	$\pi/2 > \pi - \gamma > 0$
$\pi/2 > \delta > 0$	$2(\lfloor \frac{1}{2} \lceil \frac{m}{n} \rceil \rfloor) + 1$	$2 \lceil \frac{m}{n} \rceil - 1$
	$2 \lceil \frac{m}{n} \rceil - 1$	$2 \lceil \frac{m}{n} \rceil - 1$ (when $n > 1$)

The above theorem has an interesting corollary on the Neumann boundary conditions. The inequality is sharp there!

Theorem 2. *For the regular Sturm-Liouville system (1) and (2) with Neumann boundary conditions ($\delta = \gamma = \pi/2$), if $q \geq 0$ and $0 < k \leq pw \leq K$, then for any integers $m > n \geq 1$,*

$$\frac{\mu_m}{\mu_n} \leq \{2(\lfloor \frac{1}{2} \lceil \frac{m}{n} \rceil \rfloor) + 1\}^2 \frac{K}{k},$$

with equality when and only when m is an odd multiple of n , $q \equiv 0$, and $k = K$.

We note that for any integer l , the function

$$2(\lfloor \frac{l}{2} \rfloor) + 1 = \begin{cases} l & \text{when } l \text{ is odd,} \\ l + 1 & \text{when } l \text{ is even,} \end{cases}$$

so that $\lceil m/n \rceil$ is odd when and only when

$$2(\lfloor \frac{1}{2} \lceil \frac{m}{n} \rceil \rfloor) + 1 = \lceil \frac{m}{n} \rceil.$$

Thus Theorem 2 is analogous to Ashbaugh-Benguria's result for Dirichlet boundary conditions above.

At the end of this section, we shall give an elementary proof of a trigonometric inequality in [2, 3]. This inequality is interesting itself and is critical in the method.

In section 2, we introduce the modified Prüfer substitution and the Comparison Theorem. The function $\tan^{-1}(s \tan \theta)$ is also analyzed there. The main theorems are proved in section 3. In section 4, we discuss the sign of μ_1 and obtain an upper estimate for μ_m/μ_1 for $0 < \delta, \pi - \gamma < \pi/2$ when the potential function q is sufficiently large.

Theorem 3. *Let $c > 1$. Then for $0 < \theta < \lfloor c \rfloor \pi / c$,*

$$|\sin c\theta| < c \sin \theta .$$

Proof. When c is integral-valued, the inequality can easily be proved by mathematical induction, using the compound angle formula for the sine function. On the other hand, since $f(x) = \sin x$ is a strictly concave function in $[0, \pi]$, we have for any $\phi \in (0, \pi)$,

$$\sin\left(\frac{\phi}{c}\right) > \frac{1}{c} \sin \phi .$$

Thus for arbitrary $c \in (1, 2)$, let $\phi = c\theta$; the above inequality implies $c \sin \theta > \sin c\theta$. Inductively, suppose the above inequality holds for any $c \in [1, j]$ for $j \geq 2$. Let $c = p + 1 \in (j, j + 1)$. It is easy to show that $|\sin c\theta| < c \sin \theta$ when $0 < p\theta < \lfloor p \rfloor \pi$. And when $\frac{\pi}{2} < \frac{\lfloor p \rfloor \pi}{p} \leq \theta < \frac{\lfloor p \rfloor + 1}{p+1} \pi$,

$$0 < (\lfloor p \rfloor + 1)\pi - (p + 1)\theta \leq \pi - \theta < \frac{\pi}{2} .$$

Therefore

$$|\sin(p + 1)\theta| \leq \sin \theta < (p + 1) \sin \theta .$$

The proof is complete. $\square$

Corollary 4. *If $c \geq 1$ and $|\theta| \leq \lfloor c \rfloor \pi / c$, then $\sin^2 c\theta \leq c^2 \sin^2 \theta$.*

2. PRELIMINARIES

We shall apply the modified Prüfer substitution as introduced in [3] to the regular Sturm-Liouville system. Let

$$\begin{aligned} y &= r(x) \sin a\sqrt{\mu}\theta , \\ py' &= a\sqrt{\mu}r(x) \cos a\sqrt{\mu}\theta . \end{aligned}$$

Then

$$r^2 = y^2 + \frac{(py')^2}{a^2\mu} \quad \text{and} \quad \tan a\sqrt{\mu}\theta = \frac{a\sqrt{\mu}y}{py'} .$$

Also from (1), we derive

$$\begin{aligned} (4) \quad \frac{d\theta}{dx} &= \frac{1}{p} - \left(\frac{1}{p} - \frac{w}{a^2}\right) \sin^2 a\sqrt{\mu}\theta - \frac{q}{a^2\mu} \sin^2 a\sqrt{\mu}\theta \\ &= \Phi_{(a,\mu)}(x, \theta) . \end{aligned}$$

Eq. (4) represents different first-order equations for different values of a and μ . Moreover, suppose y_m is the eigenfunction associated with the eigenvalue μ_m ; the corresponding Eq. (4) has boundary values

$$(5) \quad \begin{cases} a\sqrt{\mu_m}\theta(0) &= \tan^{-1}(a\sqrt{\mu_m} \tan \gamma), & 0 \leq \gamma < \pi , \\ a\sqrt{\mu_m}\theta(1) &= \tan^{-1}(a\sqrt{\mu_m} \tan \delta) + m\pi, & 0 < \delta \leq \pi . \end{cases}$$

The inverse function $\tan^{-1}$ takes value in $[0, \pi]$ instead of the usual $(-\frac{\pi}{2}, \frac{\pi}{2})$. In particular, let $\tan^{-1}(a\sqrt{\mu}\tan\phi) = \phi$ where $\phi = 0, \frac{\pi}{2}, \pi$.

Lemma 5. Fix any $\gamma \in [0, \pi)$; the function f defined as

$$f_\gamma(s) = \frac{1}{s}[\tan^{-1}(s \tan \gamma) - \pi]$$

is strictly increasing in $s \in \mathbf{R}^+$.

Proof. First $f_0(s) = -\frac{\pi}{s}$ and $f_{\frac{\pi}{2}}(s) = -\frac{\pi}{2s}$. Both functions are strictly increasing. For the other f_γ 's,

$$f'_\gamma(s) = \frac{s \tan \gamma - [\tan^{-1}(s \tan \gamma) - \pi](1 + s^2 \tan^2 \gamma)}{s^2(1 + s^2 \tan^2 \gamma)}.$$

Let $\varphi = \tan^{-1}(s \tan \gamma) \in [0, \pi)$. Then the numerator of f'_γ is given by

$$u(\varphi) = \tan \varphi + (\pi - \varphi)(1 + \tan^2 \varphi).$$

Since u is positive for $0 < \gamma < \frac{\pi}{2}$, the corresponding f_γ is strictly increasing. When $\frac{\pi}{2} < \gamma < \pi$, then $\frac{\pi}{2} < \varphi < \pi$ and $\tan \varphi < 0$, and

$$u'(\varphi) = 2(\pi - \varphi) \tan \varphi \sec^2 \varphi < 0.$$

Since $u(\pi) = 0$, it follows that $u(\varphi) > 0$ for $\frac{\pi}{2} < \varphi < \pi$. Therefore $f'_\gamma > 0$ and f_γ is strictly increasing. $\square$

Corollary 6. For any $\varphi \in [0, \pi]$ and $j \in \mathbf{N}$, if $1 \leq s \leq j$, then

$$\tan^{-1}(s \tan \varphi) + (j - 1)\pi \geq j\varphi.$$

Proof. It is evident that the inequality holds when $\varphi = \pi$. When $\varphi \in [0, \pi)$, we have from Lemma 5 that $\tan^{-1}(s \tan \varphi) - \pi \geq s(\varphi - \pi)$. Hence

$$\tan^{-1}(s \tan \varphi) - \varphi \geq (s - 1)(\varphi - \pi).$$

And since $1 \leq s \leq j$,

$$\tan^{-1}(s \tan \varphi) - \varphi \geq (j - 1)(\varphi - \pi).$$

$\square$

Lemma 7. For any $j \in \mathbf{N}$, $c \in \mathbf{R}$, define the function

$$g(\varphi) = \frac{1}{\varphi}\{j\pi + \tan^{-1}(c \tan \varphi)\}$$

for $\varphi \in (0, \pi)$. If $1 < c \leq g(\varphi)$, then $g'(\varphi) < 0$. Hence if $1 < c \leq g(\varphi_0)$ for some $\varphi_0 \in (0, \pi)$, then for any $0 < \varphi < \varphi_0$, $c \leq g(\varphi_0) < g(\varphi)$.

Proof. For $\varphi \in (0, \frac{\pi}{2}) \cup (\frac{\pi}{2}, \pi]$,

$$\begin{aligned} g'(\varphi) &= \frac{-1}{\varphi^2}[j\pi + \tan^{-1}(c \tan \varphi)] + \frac{1}{\varphi}\left[\frac{c \sec^2 \varphi}{1 + c^2 \tan^2 \varphi}\right] \\ &< \frac{1}{\varphi}\left[c - \frac{1}{\varphi}(j\pi + \tan^{-1}(c \tan \varphi))\right] \\ &\leq 0. \end{aligned}$$

Also, by L'Hôpital's rule,

$$\begin{aligned} g'(\frac{\pi}{2}) &= \lim_{\phi \rightarrow \pi/2} \frac{j\pi + \tan^{-1}(c \tan \phi) - (2j+1)\phi}{\phi(\phi - \pi/2)} \\ &= -\frac{2(2j+1)}{\pi} + \frac{2}{\pi} \lim_{\phi \rightarrow \pi/2} \frac{\tan^{-1}(c \tan \phi) - \pi/2}{\phi - \pi/2} \\ &= \frac{2}{c\pi} [1 - (2j+1)c] \\ &< 0 . \end{aligned}$$

□

The modified Prüfer substitution turns the problem into a first order differential equation. Hence we can use a comparison theorem to compare the growth of solutions for two different first order initial value problems. We state the following modified version of [4, p.29] for future use.

Comparison Theorem. *Consider two differential equations on $[0, 1]$,*

$$\begin{aligned} \theta_1'(x) &= F(x, \theta_1(x)) , \\ \theta_2'(x) &= G(x, \theta_2(x)) . \end{aligned}$$

Suppose F or G is Lipschitz in θ , and $F(x, \theta) \leq G(x, \theta)$ on $[0, 1] \times I$ for some interval I . If $\theta_1(0) \leq \theta_2(0)$ and $\theta_2(x)$ lies in the interior of I for every x in $(0, 1)$, then $\theta_1 \leq \theta_2$ on $[0, 1]$. In fact, take any $x_0 \in [0, 1]$, either $\theta_1(x_0) < \theta_2(x_0)$ or $\theta_1 = \theta_2$ on $[0, x_0]$.

3. PROOF OF MAIN THEOREMS

Theorem 8. *For the regular Sturm-Liouville system (1)–(3), if $q \geq 0$, $0 < k \leq pw \leq K$ and $\mu_1 > 0$, then for any integer $m \geq 1$,*

(a)

$$\frac{\mu_m}{\mu_1} \leq \left(\frac{\lfloor m/2 \rfloor \pi + \phi_{k\mu_m}}{\phi_{K\mu_1}} \right)^2 \frac{K}{k} ,$$

where

$$\phi = \min\{\delta, \pi - \gamma\} , \quad \phi_s = \tan^{-1}(\sqrt{s} \tan \phi) .$$

(b)

$$(6) \quad \frac{\mu_m}{\mu_1} < \left(\frac{(m-1)\pi + \varphi_{k\mu_m}}{\varphi_{K\mu_1}} \right)^2 \frac{K}{k} ,$$

where

$$\varphi = \max\{\delta, \pi - \gamma\} , \quad \text{and} \quad \varphi_s = \tan^{-1}(\sqrt{s} \tan \varphi) .$$

Proof. For (a), we actually need to show

$$(7) \quad \frac{\mu_m}{\mu_1} \leq \left(\max\left\{ \frac{\lfloor m/2 \rfloor \pi + \delta_{k\mu_m}}{\delta_{K\mu_1}}, \frac{(\lfloor m/2 \rfloor + 1)\pi - \gamma_{k\mu_m}}{\pi - \gamma_{K\mu_1}} \right\} \right)^2 \frac{K}{k}$$

($\gamma_s = \tan^{-1}(\sqrt{s} \tan \gamma)$ and $\delta_s = \tan^{-1}(\sqrt{s} \tan \delta)$). The two choices arise because the eigenvalues μ_m for (1) are invariant under the transformation $x \mapsto 1 - x$, while the phases at the boundary become $\pi - \delta$ and $\pi - \gamma$. Then Lemma 7 asserts

that the choice of the maximum is equivalent to the minimum of the alternative phases (i.e. $\min\{\delta_{K\mu_1}, \pi - \gamma_{K\mu_1}\}$). Hence we may simply assume that

$$(8) \quad \delta_{K\mu_1} \leq \pi - \gamma_{K\mu_1} \quad \text{or} \quad \delta \leq \pi - \gamma$$

and show that

$$\frac{\mu_m}{\mu_1} \leq \left(\frac{\lfloor m/2 \rfloor \pi + \delta_{k\mu_m}}{\delta_{K\mu_1}} \right)^2 \frac{K}{k}.$$

We compare two different equations of form (4),

$$\frac{d\theta_j}{dx} = F_j(x, \theta_j) \quad (j = 1, m),$$

where

$$\begin{cases} F_1(x, \theta) = \Phi_{\sqrt{K}, \mu_1}(x, \theta), \\ F_m(x, \theta) = \Phi_{\sqrt{k}, \mu_m}(x, \theta). \end{cases}$$

We let θ_j ($j = 1, m$) denote the modified phase function of the j th eigenfunction, with initial values

$$\begin{aligned} \sqrt{K\mu_1} \theta_1(0) &= \tan^{-1}(\sqrt{K\mu_1} \tan \gamma) - \pi \\ &= \gamma_{K\mu_1} - \pi, \end{aligned}$$

$$\begin{aligned} \sqrt{k\mu_m} \theta_m(0) &= \tan^{-1}(\sqrt{k\mu_m} \tan \gamma) - \lfloor \frac{m+1}{2} \rfloor \pi \\ &= \gamma_{k\mu_m} - \lfloor \frac{m+1}{2} \rfloor \pi. \end{aligned}$$

Hence

$$\theta_1(1) = \frac{\delta_{K\mu_1}}{\sqrt{K\mu_1}}, \quad \theta_m(1) = \frac{\lfloor m/2 \rfloor \pi + \delta_{k\mu_m}}{\sqrt{k\mu_m}}.$$

Assume

$$(9) \quad \frac{\mu_m}{\mu_1} > \left(\frac{\lfloor m/2 \rfloor \pi + \delta_{k\mu_m}}{\delta_{K\mu_1}} \right)^2 \frac{K}{k} \quad \text{for some } m > 1.$$

By (7), (8), and (9),

$$c \equiv \sqrt{\frac{k\mu_m}{K\mu_1}} > \frac{\lfloor \frac{m+1}{2} \rfloor \pi - \gamma_{k\mu_m}}{\pi - \gamma_{K\mu_1}},$$

and thus $\theta_1(0) < \theta_m(0)$. Then from $k\mu_m > K\mu_1$ and Corollary 4,

$$\frac{\sin^2 \sqrt{K\mu_1} \theta}{K\mu_1} \leq \frac{\sin^2 \sqrt{k\mu_m} \theta}{k\mu_m}$$

for $\theta \in I = [-A, A]$, $A = \frac{\lfloor c \rfloor \pi}{\sqrt{k\mu_m}}$. Hence $F_1(x, \theta) \leq F_m(x, \theta)$ for fixed $x \in [0, 1]$ and $\theta \in I$. Therefore, if $\lfloor c \rfloor \geq \lfloor m/2 \rfloor + 1$, then $\theta_m(x)$ lies in $(-A, A)$ for all $x \in (0, 1)$. It follows from the Comparison Theorem that $\theta_1(1) < \theta_m(1)$ and a contradiction with (9) is achieved. Indeed, if $\lfloor c \rfloor < \lfloor m/2 \rfloor + 1$, then by (9) and Corollary 6,

$$\lfloor c \rfloor \geq \lfloor \frac{\lfloor m/2 \rfloor \pi + \delta_{k\mu_m}}{\delta_{K\mu_1}} \rfloor = \lfloor \frac{\lfloor m/2 \rfloor \pi + \tan^{-1}(c \tan \delta_{K\mu_1})}{\delta_{K\mu_1}} \rfloor \geq \lfloor m/2 \rfloor + 1,$$

a contradiction. The proof of (a) is complete.

To prove (b), we compare the differential equations $\theta'_j(x) = F_j(x, \theta_j)$ ($j = 1, m$) on $(0, 1)$ with another set of initial values

$$\theta_1(0) = \frac{\gamma K \mu_1 - \pi}{\sqrt{K \mu_1}} \quad \text{and} \quad \theta_m(0) = \frac{\gamma k \mu_m - \pi}{\sqrt{k \mu_m}} .$$

It follows that

$$\theta_1(1) = \frac{\delta_{K \mu_1}}{\sqrt{K \mu_1}} \quad \text{and} \quad \theta_m(1) = \frac{(m-1)\pi + \delta_{k \mu_m}}{\sqrt{k \mu_m}} .$$

Assume

$$(10) \quad \frac{\mu_m}{\mu_1} \geq \left(\frac{(m-1)\pi + \delta_{k \mu_m}}{\delta_{K \mu_1}} \right)^2 \frac{K}{k} \quad \text{for some } m > 1.$$

Owing to Lemma 5, $\theta_1(0) < \theta_m(0)$. Thus $\theta_1(1) < \theta_m(1)$ if $\lfloor c \rfloor \geq m$. Indeed, if $\lfloor c \rfloor < m$, by Corollary 6 and (10),

$$\lfloor c \rfloor \geq \left\lfloor \frac{(m-1)\pi + \delta_{k \mu_m}}{\delta_{K \mu_1}} \right\rfloor = \left\lfloor \frac{(m-1)\pi + \tan^{-1}(c \tan \delta_{K \mu_1})}{\delta_{K \mu_1}} \right\rfloor \geq m ,$$

a contradiction. Hence half of (6) is proved. The transformation $x \mapsto 1 - x$ produces the rest. Therefore (b) is also valid. $\square$

Theorem 9. *Suppose all the conditions in Theorem 8 are satisfied.*

(a) *If $\pi/2 \leq \min\{\delta, \pi - \gamma\}$, then for all $m > n \geq 1$,*

$$\frac{\mu_m}{\mu_n} \leq (2 \lfloor \frac{1}{2} \lceil \frac{m}{n} \rceil \rfloor + 1)^2 \frac{K}{k} .$$

(b) *If $\pi/2 > \max\{\delta, \pi - \gamma\}$, then for all $m > n > 1$,*

$$\frac{\mu_m}{\mu_n} < (2 \lceil \frac{m}{n} \rceil - 1)^2 \frac{K}{k} .$$

Proof. We follow the method in [2]. First assume that n divides m . Use mathematical induction in n . The case $n = 1$ follows from Theorem 8. Suppose part (a) holds from $m = hn$ ($h \in \mathbf{N}$). Fix $i \in N$; for each $j < i$, let $N_j(\mu_i)$ denote the point in $(0, 1)$ such that the phase function of the i th eigenfunction first attains $\pi/2 \pmod{\pi}$ after j zeros in $(0, 1)$. Let $\omega_1 = N_1(\mu_{n+1})$ and $\omega_2 = N_h(\mu_{h(n+1)})$. If $\omega_1 \leq \omega_2$, we consider the same regular Sturm-Liouville system on $(0, \omega_1)$. Now for each $h \in \mathbf{N}$, suppose $\widetilde{\mu}_h$ is the h th eigenvalue. Then $\mu_{h(n+1)} \leq \widetilde{\mu}_h$. By Theorem 8 and Lemma 7,

$$(11) \quad \frac{\mu_{h(n+1)}}{\mu_{n+1}} \leq \frac{\widetilde{\mu}_h}{\widetilde{\mu}_1} \leq (2 \lfloor \frac{h}{2} \rfloor + 1)^2 \frac{K}{k} .$$

On the other hand, if $\omega_1 > \omega_2$, we perform the transformation $t = 1 - x$ and consider the regular Sturm-Liouville system on $(0, 1 - \omega_1)$ with a new separated boundary condition

$$\begin{aligned} y(0) \cos \delta - p(0) y'(0) \sin \delta &= 0 , \\ y'(1 - \omega_1) &= 0 . \end{aligned}$$

Therefore

$$(12) \quad \frac{\mu_{h(n+1)}}{\mu_{n+1}} \leq \frac{\widetilde{\mu}_{hn}}{\widetilde{\mu}_n} \leq (2 \lfloor \frac{h}{2} \rfloor + 1)^2 \frac{K}{k}$$

by the induction hypothesis. In general let $h = \lceil m/n \rceil$. The proof for (b) is similar. $\square$

Proof of Theorem 1. In view of Proposition 10 below, it suffices to consider the case when $0 < \delta, \pi - \gamma < \pi/2$. Here $\mu_n > 0$ for $n \geq 2$. We apply the method in Theorem 9 to this case. Since the phase of one of the boundary points attains $\pi/2 \pmod{\pi}$, Theorem 8 asserts that Eqs. (11) and (12) are still valid. $\square$

Proof of Theorem 2. The inequality follows from Theorems 8 and 9. When equality holds and $n = 1$, then $\theta_1(1) = \theta_m(1)$. By the Comparison Theorem, $\theta_1(0) = \theta_m(0)$ and $F_1(x, \theta) = F_m(x, \theta)$. So $q \equiv 0$ and $k = K$. Moreover, since $\theta_1(x) = \theta_m(x)$ at $x = 0, 1$,

$$\sqrt{\frac{k\mu_m}{K\mu_1}} = 2\lfloor \frac{m+1}{2} \rfloor - 1 = 2\lfloor \frac{m}{2} \rfloor + 1,$$

which implies that m has to be odd. The converse is trivial. Then we use induction on n . The proof is similar to [2, Proposition 3.2] and will be omitted. $\square$

4. SIGN OF μ_1 AND OTHER CASES

While it is well known that when $q \geq 0$, μ_2 is always positive (cf. [4, p. 318]), the sign of μ_1 seems to be undetermined. In this section we show that when $q \geq 0$, μ_1 is positive except possibly when $\delta, \pi - \gamma < \pi/2$. In the latter case, positive μ_1 is guaranteed when q is sufficiently large. An upper estimate for μ_m/μ_1 immediately follows.

After the transformation (see [3])

$$\frac{dt}{dx} = \frac{1}{p(x)}, \quad z(t) = y(x),$$

the regular Sturm-Liouville system as given in (1)–(3) becomes another regular Sturm-Liouville system on $(0, t_0)$,

$$(13) \quad -z''(t) + p(x(t))q(x(t))z(t) = \mu p(x(t))w(x(t))z(t),$$

satisfying

$$\begin{cases} z(0) \cos \gamma - z'(0) \sin \gamma = 0, \\ z(t_0) \cos \delta - z'(t_0) \sin \delta = 0, \end{cases}$$

where $t_0 = \int_0^1 p(x)^{-1} dx$. Applying classical Prüfer substitution on Eq. (13),

$$z(t) = r(t) \sin \phi(t), \quad z'(t) = r(t) \cos \phi(t),$$

we have (cf. [4]) the following phase equation:

$$(14) \quad \begin{cases} \phi'(t) &= (\mu p w - p q) \sin^2 \phi(t) + \cos^2 \phi(t), \\ \phi(0) &= \gamma. \end{cases}$$

If $\mu_1 \leq 0$, then we compare Eqs. (13) and (14) with the differential equation (assuming $q \geq 0$),

$$(15) \quad \hat{z}'' = 0,$$

and its phase equation

$$\begin{cases} \hat{\phi}'(t) &= \cos^2 \hat{\phi}(t), \\ \hat{\phi}(0) &= \gamma. \end{cases}$$

Obviously Eq. (15) has a solution $\hat{z}(t) = c_1 t + c_2$ where c_1, c_2 are constants satisfying

$$(16) \quad c_2 \cos \gamma - c_1 \sin \gamma = 0.$$

From the Comparison Theorem, $\phi(t) \leq \hat{\phi}(t)$ for all $t \geq 0$. We note that any solution of (14) has a zero in $(0, 1)$ iff $\phi(t_1) = \pi$ for some $t_1 \in (0, t_0)$. Thus $\hat{\phi}(t_1) \geq \pi$. But the linear function $\hat{z}(t)$ has a zero in $(0, t_0)$ iff $(c_1 t_0 + c_2)c_2 < 0$ or $\frac{c_1}{c_2} t_0 + 1 < 0$. Hence by (16), $-t_0 < \tan \gamma < 0$. The transformation $t \mapsto t_0 - t$ also gives $0 < \tan \delta < t_0$. Thus we have proved the following proposition.

Proposition 10. *If $q \geq 0$, then the eigenvalues μ_1 of a regular Sturm-Liouville system is positive, except possibly when*

$$-\int_0^1 p(x)^{-1} dx < \tan \gamma < 0 \quad \text{and} \quad 0 < \tan \delta < \int_0^1 p(x)^{-1} dx,$$

which imply $0 < \delta, \pi - \gamma < \pi/2$.

Theorem 11. *Suppose the regular Sturm-Liouville system (1)–(3) satisfies $0 < k \leq pw \leq K$. If $\varphi = \max\{\delta, \pi - \gamma\} < \pi/2$ and $b \equiv \min_{[0,1]} pq > \cot^2 \varphi$, then $\mu_1 > 0$ and*

$$\frac{\mu_m}{\mu_1} \leq \left(m - \frac{1}{2}\right)^2 \frac{\pi^2 K}{\zeta^2 k},$$

where

$$\zeta = \tan^{-1}[b \tan \varphi - \cot \varphi].$$

Proof. In view of Theorem 8(b), we only need to prove that $\varphi_{K\mu_1} \geq \zeta$, i.e. $K\mu_1 > b - \cot^2 \varphi$. WLOG, let $\varphi = \pi - \gamma$. Hence it suffices to show that $K\mu_1 > b - \cot^2 \gamma$. Let $\xi = K\mu_1 - b$. We compare Eq. (13) with

$$-\tilde{z}'' = \xi \tilde{z}.$$

Then the phase equation becomes

$$\begin{cases} \frac{d\tilde{\phi}}{dt} = \xi \sin^2 \tilde{\phi}(t) + \cos^2 \tilde{\phi}(t), \\ \tilde{\phi}(0) = \gamma > \pi/2. \end{cases}$$

So $\phi(t) \leq \tilde{\phi}(t)$ for all $t \geq 0$. If $\xi < \cot^2 \gamma$, then $\tilde{\phi}'(t) < 0$ whenever $\tilde{\phi}(t) = \gamma$. On the other hand, $\tilde{\phi}'(t) > 0$ whenever $\tilde{\phi}(t) = 0$. It follows that $\tilde{\phi}(t)$ lies in $(0, \gamma]$ and never attains π , which means that $\tilde{z}$ has no zero in $(0, \infty)$. This result contradicts the fact that z has a zero in $(0, t_0)$. $\square$

Our method of translating the phase function to the negative real axis works only when the eigenfunction has at least a zero in $(0, 1)$. The method fails when we consider the zeroth eigenvalue μ_0 . However, if one boundary condition is a Dirichlet one (say, $\gamma = 0$), then a straightforward use of the Ashbaugh-Benguria method would give

$$\frac{\mu_m}{\mu_1} \leq \left(\frac{(m-1)\pi + \delta_{K\mu_m}}{\delta_{K\mu_1}}\right)^2 \frac{K}{k}$$

provided that $\mu_1 > 0$ and arrive at an explicit upper estimate when $\delta \geq \pi/2$.

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