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UNIQUENESS FOR NON-HARMONIC TRIGONOMETRIC SERIES

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ABSTRACT. When $\lambda_n > 0$, $\lambda_n \uparrow \infty$ and

$$\frac{1}{2}|a_0| + \sum_{n=1}^{\infty} \frac{|a_n| + |b_n|}{\lambda_n^2} < \infty,$$

if

$$\frac{1}{2}a_0 + \sum_{n=1}^{\infty}(a_n \cos \lambda_n x + b_n \sin \lambda_n x) = 0 \quad \text{everywhere } (-\infty, \infty),$$

then

$$a_0 = a_1 = b_1 = \cdots = a_n = b_n = \cdots = 0.$$

More generalized results are given.

1. Introduction

Let $\{\lambda_n\}_n$ be a strictly increasing sequence of positive numbers such that

$$\lim_{n \to \infty} \lambda_n = \infty.$$

For example, $\lambda_n = \log(n+1)$ for $n = 1, 2, \ldots$. In this paper we shall discuss a uniqueness problem for non-harmonic trigonometric series:

$$\frac{1}{2}a_0 + \sum_{n=1}^{\infty}(a_n \cos \lambda_n x + b_n \sin \lambda_n x). \tag{1}$$

Many mathematicians have discussed some uniqueness problems for harmonic trigonometric series (see [1] and [3]).

Zygmund [2] discussed the same problem for the integral case

$$\int_0^{\infty} (c_s \cos sx + d_s \sin sx) ds,$$

where c_s and d_s are continuous.

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It is easy to see that the series (1) is zero at x and $-x$ if and only if

$$\frac{1}{2}a_0 + \sum_{n=1}^{\infty} a_n \cos \lambda_n x = 0; \tag{1-a}$$

$$\sum_{n=1}^{\infty} b_n \sin \lambda_n x = 0. \tag{1-b}$$

Using the same argument as in the proof of Theorem 2 in Section 68 of Chapter 1 of [1] (see p. 190), we can prove that if (1-a) and (1-b) hold, then

$$\lim_{h\to\infty} \frac{1}{2}a_0 + \sum_{n=1}^{\infty} a_n \cos \lambda_n x (\frac{\sin \lambda_n h}{\lambda_n h})^2 = 0; \tag{2-a}$$

$$\lim_{h\to\infty} \sum_{n=1}^{\infty} b_n \sin \lambda_n x (\frac{\sin \lambda_n h}{\lambda_n h})^2 = 0. \tag{2-b}$$

When(1-a) and (1-b) hold, the convergences of the two series

$$H_m^e(x) := \sum_{n=1}^{\infty} \frac{a_n \cos \lambda_n x}{\lambda^{2m}},$$

$$H_m^o(x) := \sum_{n=1}^{\infty} \frac{b_n \sin \lambda_n x}{\lambda_n^{2m}} \quad (m = 1, 2, \dots)$$

are certified by the following lemma.

Lemma 1. *When $\{\theta_n\}_n$ is a strictly decreasing sequence of positive numbers such that $\lim_{n\to\infty} \theta_n = 0$, if a series $\sum_{n=0}^{\infty} \alpha_n$ converges, then the series $\sum_{n=0}^{\infty} \alpha_n \theta_n$ converges.*

Proof. Put $R_n := \sum_{k=n}^{\infty} \alpha_k$ for $n = 1, 2, \dots$. By the Abel transform, we have

$$\sum_{k=n}^{\infty} \alpha_k \theta_k = R_n \theta_n - \sum_{k=n}^{N-1} R_{k+1}(\theta_k - \theta_{k+1}) - R_{N+1}\theta_N.$$

Thus,

$$|\sum_{k=n}^{N} \alpha_k \theta_k| \le |R_n \theta_n| + \sup_{n\le k} |R_{k+1}|(\theta_n + \theta_N) + |R_{N+1}|\theta_N$$

and each term in the right-hand side tends to zero when n and N tend to infinity. Hence the sequence $\{\sum_{k=1}^{n} \alpha_k \theta_k\}_n$ is a Cauchy sequence. The lemma is proved. □

In this paper, we shall give the following results:

Theorem 2. *If*

$$\frac{1}{2}a_0 + \sum_{n=1}^{\infty} (a_n \cos \lambda_n x + b_n \sin \lambda_n x) = 0 \quad \textit{everywhere } (-\infty, \infty), \tag{3}$$

and if for $m = 1, 2, \dots$

$$H_m^e(x) \text{ and } H_m^o(x) \text{ are continuous;} \tag{4-a}$$

$$\lim_{x\to\infty} \frac{H_m^e(x)}{x^2} = \lim_{x\to\infty} \frac{H_m^o(x)}{x} = 0, \tag{4-b}$$

then

$$a_0 = a_1 = b_1 = \cdots = a_n = b_n = \cdots = 0. \tag{5}$$

Corollary 3. *When*

$$\sum_{n=1}^{\infty} \frac{|a_n| + |b_n|}{\lambda_n^2} < \infty, \tag{6}$$

if (3) holds, then (5) is valid.

Theorem 4. *When E is an enumerable set (without loss of generality, we can assume that E satisfies $-x \in E$ if $x \in E$), if*

$$\frac{1}{2}a_n + \sum_{n=1}^{\infty}(a_n \cos \lambda_n x + b_n \sin \lambda_n x) = 0 \quad \text{everywhere } (-\infty, \infty) \text{ except } E, \tag{7}$$

and if

$$H_1^e(x) \text{ and } H_1^o \text{ are smooth in } E \tag{8}$$

and (4-a) and (4-b) hold for $m = 1, 2, \dots,$ then (5) is valid.

Corollary 5. *When*

$$\sum_{n=1}^{\infty} \frac{|a_n| + |b_n|}{\lambda_n} < \infty, \tag{9}$$

if (7) holds, then (5) is valid.

2. Proof of Theorem 2

Put

$$F_1(x) := \frac{1}{4}a_0 x^2 - H_1^e(x).$$

Thus we have

$$\frac{1}{4h^2}\{F_1(x+2h) - 2F_1(x) + F_1(x-2h)\} = \frac{1}{2}a_0 + \sum_{n=1}^{\infty} a_n \cos \lambda_n x (\frac{\sin \lambda_n h}{\lambda_n h})^2$$

and

$$\frac{1}{4h^2}\{H_1^o(x+2h) - 2H_1^o(x) + H_1^o(x-2h)\} = \sum_{n=1}^{\infty} b_n \sin \lambda_n x (\frac{\sin \lambda_n h}{\lambda_n h})^2.$$

From (3), the two second symmetric derivatives satisfy

$$D^2F_1(x) = D^2H_1^o(x) = 0 \quad \text{everywhere } (-\infty, \infty). \tag{10}$$

From (4-a) and by Lemma (3.4) in Section 3 of Chapter IX of [3] (see p. 327), $F_1(x)$ and H_1^o are linear, that is,

$$F_1(x) = A_1x + \frac{1}{2}B_1; \tag{11-a}$$

$$H_1^o(x) = C_1x + D_1 \quad \text{everywhere } (-\infty, \infty). \tag{11-b}$$

From (11-a),

$$A_1x = \frac{1}{4}a_0x^2 - \frac{1}{2}B_1 - H_1^e(x),$$

where the left-hand side is an odd function and the right-hand side is an even function. Thus, $A_1 = 0$. And from (4-b), $a_0 = 0$. Thus

$$\frac{1}{2}B_1 + H_1^e(x) = 0 \quad \text{everywhere } (-\infty, \infty). \tag{12-a}$$

Arguing analogously, for $H_1^o(x)$, we can prove that $C_1 = D_1 = 0$ and

$$H_1^o = 0 \quad \text{everywhere } (-\infty, \infty). \tag{12-b}$$

Let us discuss similarly to the above for non-harmonic trigonometric series, (12-a) and (12-b). And we can prove that $B_1 = 0$ and for some B_2

$$\frac{1}{2}B_2 + H_2^e(x) = 0;$$
$$H_2^o(x) = 0 \quad \text{everywhere } (-\infty, \infty).$$

Continuing this process, we have $B_{m-1} = 0$ and for some B_m

$$\frac{1}{2}B_m + H_m^e(x) = 0; \tag{13-a}$$

$$H_m^o(x) = 0 \quad \text{everywhere } (-\infty, \infty). \tag{13-b}$$

Obviously $B_m = 0$ for all m; then

$$H_m^e(x) = 0 \quad \text{everywhere } (-\infty, \infty). \tag{13-a$'$}$$

The conclusion follows if the following lemma is proved.

Lemma 6. *Let $\{\theta_n\}_n$ be a sequence satisfying the condition of Lemma 1 and $\theta_1 < 1$. If $\sum_{n=1}^{\infty} \alpha_n$ converges and*

$$\alpha_0 + \sum_{n+1}^{\infty} \alpha_n\theta_n^m = 0 \quad \textit{for all } m = 1, 2, \ldots,$$

then $\alpha_0 = 0$.

Proof of Lemma 6. Put $R_n = \sum_{k=n}^{\infty} \alpha_k$ for $n = 1, 2, \ldots$. For each $\varepsilon > 0$, there exists N such that $|R_n| < \varepsilon$ for $n > N$. Since

$$\begin{aligned}\alpha_0 + \sum_{n=1}^{\infty} \alpha_n \theta_n^m = \alpha_0 + \alpha_1 \theta_1^m + \cdots + \alpha_N \theta_N^m \\ -R_{N+1}\theta_{N+1}^m - \sum_{N+1}^{\infty} R_{k+1}(\theta_k^m - \theta_{k+1}^m),\end{aligned}$$

$$|R_{N+1}\theta_{N+1}^m| < \varepsilon,$$

$$\begin{aligned}\left|\sum_{k=N+1}^{\infty} R_{k+1}(\theta_k^m - \theta_{k+1}^m)\right| &\le \sum_{k=N+1}^{\infty} |R_{k+1}||\theta_k^m - \theta_{k+1}^m| \\ &\le \varepsilon \sum_{k=N+1}^{\infty} \theta_k^m - \theta_{k+1}^m = \varepsilon\theta_{N+1}^m < \varepsilon\end{aligned}$$

and

$$\lim_{m\to\infty} \sum_{k=1}^{N} \alpha_k \theta_k^m = 0,$$

we have

$$|\alpha_0| = \left|\sum_{k=1}^{\infty} \alpha_k \theta_k^m\right| \le \left|\sum_{k=1}^{N} \alpha_k \theta_k^m\right| + \left|\sum_{k=N+1}^{\infty} \alpha_k \theta_k^m\right| \le \left|\sum_{k=1}^{N} \alpha_k \theta_k^m\right| + 2\varepsilon.$$

Thus

$$|\alpha_0| \le \lim_{m\to\infty} \left|\sum_{k=1}^{N} \alpha_k \theta_k^m\right| + 2\varepsilon = 2\varepsilon.$$

Consequently $\alpha_0 = 0$. Lemma 6 is proved. □

Now put $\theta_n = \frac{\lambda_{n+1}}{\lambda_1}$ for $n = 1, 2, \ldots$. Thus $\{\theta_n\}_n$ satisfies the condition of Lemma 6. And put $\alpha_n = a_{n+1} \cos \lambda_{n+1} x$. Then from (13-a′)

$$a_1 \cos \lambda_1 x = 0 \quad \text{everywhere } (-\infty, \infty).$$

And analogously from (13-b),

$$b_1 \sin \lambda_n x = 0 \quad \text{everywhere } (-\infty, \infty).$$

Continuing this process we can easily prove

$$a_n \cos \lambda_n x = b_n \sin \lambda_n x = 0 \quad \text{everywhere for all } n.$$

Consequently

$$a_n = b_n = 0 \quad \text{for } n = 1, 2, \ldots.$$

We have proved Theorem 2.

3. Proofs of Theorem 4 and corollaries

By Lemma (3.20) in Section 3 of Chapter IX of [3] (see p.328) and from (7), (8) and (4-a), $F_1(x)$ and $H_1^o(x)$ are linear. Then using the same argument as in the proof of Theorem 2, we can easily prove Theorem 4.

Obviously condition (6) in Corollary 3 is stronger than (4-a) and (4-b) in Theorem 2, and (9) in Corollary 5 is stronger than (4-a), (4-b) and (8) in Theorem 4.

Remark 1. Under the conditions (4-a) and (4-b), $H_m^e(x)$ and $H_m^o(x)$ are continuous if and only if

$$\lim_{h\to 0}\sum_{n=1}^{\infty}\frac{a_n\sin\lambda_n x}{\lambda_n^{2m}}\sin\lambda_n h=\lim_{h\to 0}\sum_{n=1}^{\infty}\frac{b_n\cos\lambda_n x}{\lambda_n^{2m}}\sin\lambda_n h=0$$

$$\text{everywhere } (-\infty,\infty).$$

Remark 2. $H_1^e(x)$ and $H_1^o(x)$ are smooth at x if and only if

$$\lim_{h\to 0}h(\frac{1}{2}a_0+\sum_{n=1}^{\infty}a_n\cos\lambda_n x(\frac{\sin\lambda_n h}{\lambda_n h})^2)=\lim_{h\to 0}h(\sum_{n=1}^{\infty}b_n\sin\lambda_n x(\frac{\sin\lambda_n h}{\lambda_n h})^2)=0.$$

(See p. 43 (3.1) in Chapter II and p. 328 (3.21) in Chapter IX of [3].)

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