VANISHING OF THE LEADING TERM IN HARISH-CHANDRA'S LOCAL CHARACTER EXPANSION

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ABSTRACT. Harish-Chandra's formula for the character Θ_π of an irreducible smooth representation π of a reductive p -adic group G expresses Θ_π near 1 as a linear combination of the Fourier transforms of nilpotent G -orbits in the Lie algebra of G . In this note, we prove that if π is tempered but not in the discrete series, then the coefficient attached to the zero nilpotent orbit vanishes.

1. INTRODUCTION

Let $\mathbf{G}$ be a connected, reductive algebraic group defined over a p -adic field F , and let $G = \mathbf{G}(F)$. A smooth irreducible representation of G has a character, which after Harish-Chandra [3, 4] is known to be a locally integrable function on G , locally constant on the open subset of regular elements. (See the first two sections of [15] for basic definitions.) Irreducible characters are, of course, basic objects in harmonic analysis on the locally compact group G . Since irreducible representations of G are almost always infinite-dimensional, the behavior of their characters near the identity in G is especially interesting. A deep result of Harish-Chandra [4] (and earlier Howe [7] for the general linear group) addresses this topic. Let $\mathcal{O}$ be a nilpotent G -orbit in $\mathfrak{g}$; then integration over $\mathcal{O}$ defines a G -invariant Radon measure $\mu_{\mathcal{O}}$ on $\mathfrak{g}$ [13]. These are the nilpotent orbital integrals; their Fourier transforms $\hat{\mu}_{\mathcal{O}}$ (taken relative to the additive group of $\mathfrak{g}$) are in fact locally integrable functions on $\mathfrak{g}$ which are locally constant on the open subset of regular elements [4]. Let π be a smooth irreducible (or just finite length) representation of G , and let Θ_π be its character. Harish-Chandra's result asserts that there exist complex numbers $c_{\mathcal{O}}(\pi)$ such that

$$(1) \quad \Theta_\pi(\exp X) = \sum_{\mathcal{O}} c_{\mathcal{O}}(\pi) \hat{\mu}_{\mathcal{O}}(X)$$

where $\mathcal{O}$ runs through nilpotent orbits in $\mathfrak{g}$ and X is a regular element sufficiently close to 0. We note that (i) the exponential map is defined and invertible in a neighborhood of 0 in $\mathfrak{g}$; and (ii) there are only finitely many nilpotent orbits, so the sum above is finite. (Both statements fail in positive characteristic.)

Little is known in general about the $\hat{\mu}_{\mathcal{O}}$'s. In case $\mathcal{O}$ is a stable Richardson orbit, the argument of Howe [7] shows that $\hat{\mu}_{\mathcal{O}}$ coincides near 1 with the induced character arising from the trivial representation of the parabolic subgroup corresponding to

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$\mathcal{O}$. (See [9] for a generalization.) They can therefore be computed by the formula of Van Dijk [18]. Assem [1] has determined the $\hat{\mu}_{\mathcal{O}}$'s for $SL_{\ell}(F)$, where ℓ is prime.

An important property of the $\hat{\mu}_{\mathcal{O}}$'s is homogeneity: if $d(\mathcal{O})$ denotes the dimension of the nilpotent orbit $\mathcal{O}$, then we have $\hat{\mu}_{\mathcal{O}}(t^2X) = |t|^{-d(\mathcal{O})} \hat{\mu}_{\mathcal{O}}(X)$ for X regular and $t \in F^{\times}$ [4]. Thus the coefficients $c_{\mathcal{O}}(\pi)$ partly determine the asymptotic properties of Θ_{π} . (Murnaghan [10, 11] has determined these coefficients for supercuspidal representations of certain rank 2 and 3 groups. Mœglin and Waldspurger [9] have generalized a result of Rodier which interprets the coefficients $c_{\mathcal{O}}(\pi)$ for $\mathcal{O}$ regular as the dimension of the space of Whittaker functionals for π . See also §2 below.)

We are interested in the coefficient $c_0(\pi)$ corresponding to the zero nilpotent orbit. Since μ_0 is the Dirac delta at $0 \in \mathfrak{g}$, we have $\hat{\mu}_0 = 1$; thus the term corresponding to the zero nilpotent orbit in (1) is just the constant $c_0(\pi)$. The purpose of this note is to prove the following

Theorem. *Let π be an irreducible tempered representation. Then $c_0(\pi) = 0$ if π is not in the discrete series.*

A regular element of G (or $\mathfrak{g}$) is called elliptic if its centralizer in G is compact modulo the center of G . An irreducible representation π is called elliptic if its character does not vanish identically on the set G^{ell} of elliptic regular elements of G . The theorem allows one to distinguish the discrete series characters from other elliptic tempered characters by their behavior on the elliptic set G^{ell} .

For a supercuspidal representation π , Harish-Chandra showed ([4], Theorem 6) that $c_0(\pi)$ equals the formal degree of π times a constant depending only on G and the choice of Haar measure on G . Rogawski determined the constant in [14]. This formula is known to hold for π in the discrete series. We use results of Kazhdan [8] to generalize this formula to arbitrary smooth irreducible π (Proposition 1 below, for the case of compact center). The theorem then follows from a simple application of Harish-Chandra's Plancherel formula (§5).

I would like to thank Paul Sally for introducing me to Harish-Chandra's theory of invariant harmonic analysis and for suggesting the result of this note.

2. PRELIMINARIES

We keep the notation of the introduction, but require that the center of G be compact. This allows us to use the results of [8] without modification. This restriction will be removed in §6.

Let $C_c^{\infty}(G)$ be the space of smooth, compactly supported functions on G . Fix a Haar measure dg on G . For $f \in C_c^{\infty}(G)$ let $f^{\vee}(g) = f(g^{-1})$. Let $\mathcal{E}(G)$ denote the set of (equivalence classes of) irreducible smooth representations of G . Define a pairing $\langle \cdot, \cdot \rangle$ between irreducible representations and functions:

$$(2) \quad \langle \pi, f \rangle = \text{trace } \pi(f^{\vee}) = \int_G \Theta_{\pi}(g) f^{\vee}(g) dg.$$

Let $\tilde{\pi}$ denote the smooth contragredient of $\pi \in \mathcal{E}(G)$. Then we have $\langle \pi, f^{\vee} \rangle = \langle \tilde{\pi}, f \rangle$. (In the next section we will attach functions $f \in C_c^{\infty}(G)$ to representations $\pi \in \mathcal{E}(G)$ (via the relation (6)). When π is supercuspidal, its matrix coefficients lie in $C_c^{\infty}(G)$, and we may take f to be a matrix coefficient f_{π} with $f_{\pi}(1) = d(\pi)$ [3, p. 92–94]. The Schur orthogonality relations [3, p. 4] then insure that $\langle \varrho, f_{\pi} \rangle = \delta_{\varrho, \pi}$ for any $\varrho \in \mathcal{E}(G)$. Had we defined the pairing (2) to be $\text{trace } \pi(f)$, the orthogonality relations would take a slightly less convenient form. See also (7) in §3.)

The Weyl integration formula [3] allows us to view $\langle \cdot, \cdot \rangle$ as a pairing between characters and orbital integrals. Let $\{T\}$ denote a complete set of representatives of conjugacy classes of Cartan subgroups in G , where by a Cartan subgroup we mean the F -points of a maximal torus of $\mathbf{G}$ defined over F . Then we have

$$(3) \quad \int_G \Theta_\pi(g) f^\vee(g) dg = \sum_{\{T\}} |W_T|^{-1} \int_{T^{\text{reg}}} |D(t)| \Theta_\pi(t) \int_{G/T} f^\vee(xtx^{-1}) d\dot{x} dt,$$

where T^{reg} denotes the regular elements in T , dt is a fixed Haar measure on T , D is the usual Weyl discriminant, and $d\dot{x}$ is the quotient measure. If T is elliptic (compact) we choose the measure dt so that the volume of T is 1.

Orbital integrals have a local expansion, due to Shalika [16], somewhat analogous to that of characters. If $\mathcal{O}$ is a unipotent orbit in G , let $\Lambda_{\mathcal{O}}$ denote integration over $\mathcal{O}$. There exist functions $\Gamma_{\mathcal{O}}^T$ indexed by unipotent orbits $\mathcal{O}$ and Cartan subgroups T with the following property. Let $f \in C_c^\infty(G)$; then

$$(4) \quad \int_{G/T} f(xtx^{-1}) d\dot{x} = \sum_{\mathcal{O}} \Gamma_{\mathcal{O}}^T(t) \Lambda_{\mathcal{O}}(f)$$

for t a regular element in T sufficiently close to the identity. The functions $\Gamma_{\mathcal{O}}^T$ are called Shalika germs. (For a relation between values of the germs $\Gamma_{\mathcal{O}}^T$ and the coefficients $c_{\mathcal{O}}$ see [12].)

The germs have homogeneity properties: if $\exp X$ is a regular element of T sufficiently close to the identity, then

$$\Gamma_{\mathcal{O}}^T(\exp t^2 X) = |t|^{-d(\mathcal{O})} \Gamma_{\mathcal{O}}^T(\exp X)$$

for $t \in F^\times$ with $|t| \leq 1$. The germ Γ_1^T associated to the trivial unipotent orbit is constant [4]. Γ_1^T vanishes unless T is elliptic, in which case Rogawski [14] has determined its value. Let ℓ denote the semisimple F -rank of G . Then

$$(5) \quad \Gamma_1^T = \frac{(-1)^\ell}{d(\mathbf{st})}$$

where T is an elliptic Cartan subgroup and $d(\mathbf{st})$ is the formal degree of the Steinberg representation $\mathbf{st}$ with respect to the Haar measure $dg = d\dot{x} dt$.

3. SOME RESULTS OF KAZHDAN

We review here some of the results of Kazhdan [8]. Let $R(G)$ be the space of formal $\mathbf{C}$ -linear combinations of irreducible representations of G . If P is a proper parabolic subgroup of G with Levi factorization $P = MN$, then (unitary) induction defines a map $R(M) \rightarrow R(G)$ (which does not depend on P , just M). Let $R_I(G)$ be the subspace spanned by the images of all these maps for proper parabolic subgroups. Let $\bar{R}(G)$ be the quotient $R(G)/R_I(G)$. Kazhdan's results imply that the image of $\pi \in \mathcal{E}(G)$ is zero if and only if π is not elliptic. (A subtlety, noted first by Herb [6], is that there exist tempered representations which are not elliptic yet are not properly induced.) The Langlands classification shows that $\bar{R}(G)$ is spanned by the classes of irreducible tempered representations.

Let $A(G) \subset C_c^\infty(G)$ consist of functions whose regular, nonelliptic orbital integrals vanish (such as matrix coefficients of supercuspidal representations [3, p. 92]). Let $J(G)$ be the subspace of functions all of whose regular orbital integrals vanish, including the elliptic ones. Kazhdan shows that $J(G)$ is the space of functions f

with trace $\pi(f) = 0$ for all $\pi \in \mathcal{E}(G)$ (equivalently, for all π tempered irreducible), and that $J(G)$ is also the commutator subspace of $C_c^\infty(G)$. Hence all invariant distributions vanish on $J(G)$. Let $\bar{A}(G) = A(G)/J(G)$.

The pairing (2) extends by linearity to a pairing between $R(G)$ and $C_c^\infty(G)$. Evidently $J(G)$ is the annihilator of $R(G)$. Since induced characters vanish on the elliptic set (see e.g. [18]), the Weyl integration formula (3) shows that $R_I(G)$ annihilates $A(G)$. Thus $\langle \cdot, \cdot \rangle$ descends to a pairing $\langle \cdot, \cdot \rangle : \bar{R}(G) \times \bar{A}(G) \rightarrow \mathbf{C}$.

The key result of [8] for us is the existence of an isomorphism $\vartheta : \bar{R}(G) \rightarrow \bar{A}(G)$ according to which $\vartheta(\pi) = \bar{f}$ means that the character of π is given on the elliptic set by the elliptic orbital integrals of f :

$$(6) \quad \int_G f(x\gamma x^{-1}) dx = \Theta_\pi(\gamma) \quad \text{for } \gamma \in G^{\text{ell}}.$$

The formula (3) shows that ϑ is self-adjoint: $\langle \pi, \vartheta(\varrho) \rangle = \langle \varrho, \vartheta(\pi) \rangle$. (In fact this expression is just the elliptic inner product between the characters Θ_ϱ and Θ_π , for in the formula (3) the nonelliptic Cartan subgroups make no contribution.) We note also that if ϱ is in the discrete series, $\vartheta(\varrho)$ is a pseudo-coefficient for ϱ in the sense that

$$(7) \quad \langle \pi, \vartheta(\varrho) \rangle = \begin{cases} 0 & \text{if } \pi \text{ is tempered, irreducible and } \pi \not\cong \varrho, \\ 1 & \text{if } \pi \cong \varrho. \end{cases}$$

Indeed, if f is a properly normalized matrix coefficient for ϱ , then $\langle \pi, f \rangle$ is given by (7), and by a theorem of Harish-Chandra (cf. [2, Proposition 5]) the regular non-elliptic orbital integrals of f vanish, while on the elliptic set we have the analogue of (6). Thus $f - \vartheta(\varrho)$ is a Schwartz function all of whose regular orbital integrals vanish; by (3) it annihilates all tempered irreducible π . Then f and $\vartheta(\varrho)$ agree on all tempered irreducible π , so the relations (7) follow.

4. A FORMULA FOR THE LEADING TERM

For π in the discrete series, the coefficient $c_0(\pi)$ is essentially the formal degree of π . The formal degree, in turn, is just $f(1)$ if f is a pseudo-coefficient for π . The following proposition generalizes this relation.

Proposition 1. *If $\pi \in \mathcal{E}(G)$ and $\vartheta(\pi) = \bar{f}$, then*

$$c_0(\pi) = \frac{(-1)^\ell}{d(\mathbf{st})} f(1).$$

Remark. The formal degree $d(\mathbf{st})$ is computed with respect to the same Haar measure that appears in (6). Changing dx to $c dx$ gives $d(\mathbf{st}, c dx) = c^{-1} d(\mathbf{st}, dx)$, and in place of $\bar{f}$ we get $c^{-1} \bar{f}$. Thus the right side is independent of choice of Haar measure, as it should be. Note also that $f(1)$ is independent of the choice of f representing $\bar{f}$.

Proof. In view of (6) we can compare the local character expansion (1) for π on the elliptic set with the Shalika germ expansion (4) for the elliptic orbital integrals of $\bar{f} = \vartheta(\pi)$. Let T be any elliptic Cartan subgroup in G . We have

$$(8) \quad \sum c_{\mathcal{O}}(\pi) \hat{\mu}_{\mathcal{O}}(\log \gamma) = \sum \Gamma_{\mathcal{O}}^T(\gamma) \Lambda_{\mathcal{O}}(f)$$

for $\gamma \in T^{\text{reg}}$ sufficiently near 1. On the left, $\mathcal{O}$ runs through nilpotent orbits in the Lie algebra $\mathfrak{g}$; on the right $\mathcal{O}$ runs through unipotent orbits in G . (The exponential

map gives a bijection between these two sets, preserving the dimension $d(\mathcal{O})$ of the orbit $\mathcal{O}$.)

Set $\gamma = \exp t^2 X$. We choose X close enough to 0 such that the exponential map is defined for all tX with $|t| \leq 1$, and (8) holds for $0 < |t| \leq 1$. Then the homogeneity properties of the $\hat{\mu}_{\mathcal{O}}$'s and the germs allow us to view equation (8) as an equation between linear combinations of “characters” $|t|^d$ of the sub-semigroup $\{t : 0 < |t| \leq 1\} \subset F^\times$. By linear independence of characters, the coefficients of distinct characters must be equal. Thus

$$\sum_{\mathcal{O}: d(\mathcal{O})=d} c_{\mathcal{O}}(\pi) \hat{\mu}_{\mathcal{O}}(X) = \sum_{\mathcal{O}: d(\mathcal{O})=d} \Gamma_{\mathcal{O}}^T(\exp X) \Lambda_{\mathcal{O}}(f).$$

When $d = 0$ the only orbits in the sum are the trivial orbits 0 and 1 respectively; hence $c_0(\pi) = \Gamma_1^T f(1)$. Now the proposition reduces to Rogawski's formula (5) for the germ Γ_1^T for the elliptic Cartan subgroup T . $\square$

Remark. If π is square-integrable, we recover the formula of Harish-Chandra and Rogawski using the fact that $f(1) = d(\pi)$. This formula, together with the main theorem, completely determines c_0 viewed as a linear functional on $\bar{R}(G)$, since, as noted in §3, $\bar{R}(G)$ is spanned by tempered irreducible representations.

5. VANISHING OF $f(1)$

It remains to see that if π is tempered but not in the discrete series, then $\bar{f} = \vartheta(\pi)$ vanishes at 1. For this, we examine the Plancherel formula [5]. Since $f \in A(G)$, its scalar-valued Fourier transform $\hat{f}$ vanishes on induced representations. Thus there is no contribution from proper Levi subgroups, and the Plancherel formula becomes completely explicit:

$$f(1) = \sum_{\varrho \in \mathcal{E}_2(G)} \hat{f}(\varrho) d(\varrho)$$

where $\mathcal{E}_2(G)$ denotes the discrete series representations.

Now $\hat{f}(\varrho) = \text{trace } \varrho(f)$ is just $\langle \pi, \vartheta(\tilde{\varrho}) \rangle$ by self-adjointness of ϑ . For $\varrho \in \mathcal{E}_2(G)$ and π as above, $\vartheta(\tilde{\varrho})$ is a pseudo-coefficient for the discrete series representation $\tilde{\varrho}$; thus by (7) we have $\hat{f}(\varrho) = 0$. Thus

Proposition 2. *Let $\pi \in \mathcal{E}(G)$ be tempered but not in the discrete series, and let f be a representative of $\bar{f} = \vartheta(\pi)$. Then $\hat{f}(\varrho) = 0$ for all $\varrho \in \mathcal{E}_2(G)$.*

Corollary. *With the notation of the proposition, $f(1) = 0$.*

The theorem (for the case where G has compact center) now follows from this corollary and the formula of Proposition 1. $\square$

6. CONCLUSION

To conclude the proof of the theorem, we show how to drop the assumption that G has compact center. Thus let $\mathbf{G}$ be as in the introduction. Let $\mathbf{G}^{\text{der}}$ be the derived subgroup of $\mathbf{G}$, and let $G' = \mathbf{G}^{\text{der}}(F)$. Then G' has compact, in fact finite, center. Let π be an irreducible, tempered, non-square-integrable representation of G . By Lemma 2.1 of [17] (valid even in positive characteristic),

$$\pi|_{G'} = \pi_1 \oplus \pi_2 \oplus \cdots \oplus \pi_r$$

where the π_i are smooth irreducible representations of G' . By Proposition 2.7 of [17] each π_i is tempered and not in the discrete series.

Since a regular elliptic element of $\mathfrak{g}'$ is also a regular elliptic element of $\mathfrak{g}$, we can compare the local character expansions of π on G and $\pi|_{G'}$ on G' as in the proof of Proposition 1 to see that $c_0(\pi|_{G'}) = c_0(\pi)$. Now the character of π restricted to G' is the sum of the characters of the π_i ; so $c_0(\pi) = c_0(\pi_1) + c_0(\pi_2) + \cdots + c_0(\pi_r) = 0 + \cdots + 0 = 0$. Thus the theorem holds without restriction on the center of G . $\square$

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