DIFFEOMORPHISMS WITH PERSISTENCY

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ABSTRACT. The C^1 interior of the set of all diffeomorphisms satisfying Lewowicz's persistency is characterized as the set of all diffeomorphisms satisfying Axiom A and the strong transversality condition.

In [5], Lewowicz introduced a notion of persistency for a homeomorphism of a compact metric space X , and it is remarked that persistence is a weaker property than topological stability when X is a manifold. It is also proved there that every pseudo-Anosov map (on a surface) is persistent. The purpose of this paper is to analyze the dynamics of diffeomorphisms having persistency. More precisely we shall prove the following theorem.

Let M be a C^∞ closed manifold and let $\text{Diff}(M)$ be the space of C^1 diffeomorphisms of M endowed with C^1 topology. We denote by $\mathcal{P}(M)$ the set of all $f \in \text{Diff}(M)$ having persistency.

Theorem. *The C^1 interior of $\mathcal{P}(M)$ in $\text{Diff}(M)$, $\text{int}\mathcal{P}(M)$, is characterized as the set of all diffeomorphisms satisfying Axiom A and the strong transversality condition.*

It was proved in [7] and [9] respectively that the C^1 interior of the set of all $f \in \text{Diff}(M)$ having topological stability and the C^1 interior of the set of all $f \in \text{Diff}(M)$ having the pseudo-orbit tracing property were characterized as the set of all diffeomorphisms satisfying Axiom A and the strong transversality condition. Therefore, if the theorem is established, then these two open sets and $\text{int}\mathcal{P}(M)$ are equal.

Let $\mathcal{E}(M)$ be the set of all expansive diffeomorphisms of M . Gerber and Katok [3] proved that if f is a pseudo-Anosov map on a surface M and if $\mathcal{N}^0(f)$ is a C^0 neighborhood of f , then there exists a smooth diffeomorphism $g \in \mathcal{N}^0(f)$ conjugating to f . Thus it can be checked that $g \in \mathcal{E}(M) \cap \mathcal{P}(M)$, and more precisely, the following corollary implies that g belongs to $\mathcal{E}(M) \cap \partial\mathcal{P}(M)$. Here $\partial\mathcal{P}(M)$ denotes the boundary of $\mathcal{P}(M)$ in $\text{Diff}(M)$.

Corollary. *$\mathcal{E}(M) \cap \text{int}\mathcal{P}(M)$ is characterized as the set of all Anosov diffeomorphisms.*

The corollary is an easy consequence of our theorem. Indeed, since every $f \in \text{Diff}(M)$ satisfying Axiom A and the strong transversality condition is structurally

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stable, if $f \in \mathcal{E}(M)$, then it is Anosov (by [6]). Conversely, if $f \in \text{Diff}(M)$ is Anosov, then f is persistent since f is topologically stable.

Let d be a metric on M which is induced from a Riemannian metric $\|\cdot\|$ on TM , and let us denote by $\mathcal{H}(M)$ the set of all homeomorphisms of M . We say that $f \in \mathcal{H}(M)$ is *persistent* if for each $\varepsilon > 0$, there is $\delta > 0$ such that for every $x \in M$ and $g \in \mathcal{H}(M)$ with $d(f, g) < \delta$, there is $y \in M$ satisfying $d(f^n(x), g^n(y)) < \varepsilon$ ($\forall n \in \mathbf{Z}$). The notion is independent of a metric for M and is conjugacy invariant.

Let $\Lambda(f)$ be a hyperbolic set of $f \in \text{Diff}(M)$. For any $\varepsilon > 0$ and $x \in \Lambda(f)$, the *local stable manifold* and the *local unstable manifold* are denoted by $W_\varepsilon^s(x, f)$ and $W_\varepsilon^u(x, f)$ respectively. The *stable manifold* $W^s(x, f)$ and the *unstable manifold* $W^u(x, f)$ of $x \in \Lambda(f)$ are defined by a usual way. Let $f \in \text{Diff}(M)$ satisfy Axiom A. Then the *non-wandering set* of f , $\Omega(f)$, is a disjoint union of basic sets $\Lambda_1(f) \cup \cdots \cup \Lambda_\ell(f)$. Recall that the periodic points of $f|_{\Lambda_i(f)}$ are dense in $\Lambda_i(f)$ ($1 \leq i \leq \ell$) and that for every $x \in M$ there are $p \in \Lambda_i(f)$ and $q \in \Lambda_j(f)$ ($1 \leq i \neq j \leq \ell$) such that $x \in W^s(p, f) \cap W^u(q, f)$. We say that f satisfies the *strong transversality condition* if for every $x \in M$, $T_x W^s(p, f) + T_x W^u(q, f) = T_x M$ for some $p, q \in \Omega(f)$.

Let $P(f)$ denote the set of all periodic points of $f \in \text{Diff}(M)$, and let $\mathcal{F}(M)$ be the set of all $f \in \text{Diff}(M)$ having a C^1 -neighborhood $\mathcal{U}(f) \subset \text{Diff}(M)$ such that every $p \in P(g)$ ($\forall g \in \mathcal{U}(f)$) is hyperbolic. Then such a set was characterized as the set of all diffeomorphisms satisfying Axiom A with no-cycles (see [1, 4]). It is well known that every $f \in \text{Diff}(M)$ satisfying Axiom A and the strong transversality condition is persistent (because f is topologically stable (see [8])). Therefore our theorem follows from the following two propositions.

Proposition A. *The C^1 interior of $\mathcal{P}(M)$, $\text{int}\mathcal{P}(M)$, is a subset of $\mathcal{F}(M)$.*

Proposition B. *Let $f \in \text{Diff}(M)$ satisfy Axiom A with no-cycles. If $f \in \text{int}\mathcal{P}(M)$, then f satisfies the strong transversality condition.*

1. PROOF OF PROPOSITION A

Let $f \in \text{int}\mathcal{P}(M)$. To get the conclusion, it is enough to show that every $p \in P(f)$ is hyperbolic. Indeed, if this is established, then for every C^1 neighborhood $\mathcal{V}(f) \subset \text{int}\mathcal{P}(M)$ of f , every $q \in P(g)$ ($\forall g \in \mathcal{V}(f)$) is hyperbolic because $g \in \text{int}\mathcal{P}(M)$. Thus $f \in \mathcal{F}(M)$ is obtained.

Fix a neighborhood $\mathcal{U}(f) \subset \text{int}\mathcal{P}(M)$ of f , and by assuming that there is a non-hyperbolic periodic point $p = f^n(p)$, we shall derive a contradiction. Here $n > 0$ is the prime period of p . The tangent space $T_p M$ splits into the direct sum $T_p M = E_p^u \oplus E_p^s \oplus E_p^c$ where E_p^u, E_p^s and E_p^c are $D_p f^n$ -invariant subspaces corresponding to the absolute values of the eigenvalues of $D_p f^n$ greater than one, less than one and equal to one, and suppose $E_p^c \neq 0$. Then, for every $\varepsilon > 0$ there exists a linear automorphism $\mathcal{O} : T_p M \rightarrow T_p M$ such that

$$\begin{cases} \|\mathcal{O} - I\| \leq \varepsilon, \\ \mathcal{O}(E_p^\sigma) = E_p^\sigma \text{ for } \sigma = s, u \text{ and } c, \\ \text{all eigenvalues of } \mathcal{O} \circ D_p f^n|_{E_p^c} \text{ are of a root of unity,} \end{cases}$$

where $I : T_p M \rightarrow T_p M$ is an identity map. By making use of Franks's lemma (see [2, Lemma 1.1]), we can find $\delta_0 > 0$ and $g \in \mathcal{U}(f)$ such that

$$(i) \quad B_{4\delta_0}(f^i(p)) \cap B_{4\delta_0}(f^j(p)) = \emptyset \text{ for } 0 \leq i \neq j \leq n-1,$$

- (ii) $g(x) = f(x)$ for $x \in \{p, f(p), \dots, f^{n-1}(p)\} \cup \{M \setminus \cup_{i=0}^{n-1} B_{4\delta_0}(f^i(p))\}$,
- (iii) $g(x) = \exp_{f^{i+1}(p)} \circ D_{f^i(p)} f \circ \exp_{f^i(p)}^{-1}(x)$ for $x \in B_{\delta_0}(f^i(p))$ ($0 \leq i \leq n-2$),
- (iv) $g(x) = \exp_p \circ \mathcal{O} \circ D_{f^{n-1}(p)} f \circ \exp_{f^{n-1}(p)}^{-1}(x)$ for $x \in B_{\delta_0}(f^{n-1}(p))$,

where $B_\varepsilon(x) = \{y \in M | d(x, y) \leq \varepsilon\}$ for $\varepsilon > 0$.

Define $G = \mathcal{O} \circ D_p f^n$. Then there exists $m > 0$ such that $G|_{E_p^c}$ is an identity map. For a sufficiently small $0 < \delta_1 < \delta_0$, we have

$$g|_{\exp_p T_p M(\delta_1)}^{mn} = \exp_p \circ G^m \circ \exp_p^{-1}$$

where $T_p M(\delta_1) = \{v \in T_p M | \|v\| \leq \delta_1\}$. Put $E_p^c(\delta_1) = E_p^c \cap T_p M(\delta_1)$. Then it is clear that

$$g|_{\exp_p E_p^c(\delta_1)}^{mn} = id|_{\exp_p E_p^c(\delta_1)}.$$

Let $v = (v_1, v_2, \dots, v_r)$ ($r = \dim E_p^c$) be the representation by components with respect to the fundamental vectors of $\mathbf{R}^r = E_p^c$. Put $\hat{\varepsilon} = \delta_1/8$ and fix any $0 < \delta \leq \hat{\varepsilon}$. Let $\varphi : \mathbf{R}^r \rightarrow \mathbf{R}^r$ be the time-one map given by the vector field

$$\dot{v}_i = \delta' \chi(v_1) \cdots \chi(v_r) v_i$$

for $1 \leq i \leq r$. Here $\chi : \mathbf{R} \rightarrow \mathbf{R}$ is a C^∞ function ($0 \leq \chi(t) \leq 1$) such that

$$\chi(t) = \begin{cases} 1 & \text{if } |t| \leq \delta_1/2, \\ 0 & \text{if } |t| \geq 2\delta_1/3, \end{cases}$$

and $\delta' > 0$ is a number chosen so that $\|\varphi(v) - v\| < \delta$ for $v \in \mathbf{R}^r$ and $\|D_v \varphi - id|_{\mathbf{R}^r}\| \leq |e^{\delta'} - 1| < \delta$ for $v \in T_p M(\delta_1) \cap \mathbf{R}^r$. We shall denote by $\tilde{\varphi} : T_p M(\delta_1) \rightarrow T_p M(\delta_1)$ the extension of φ such that $\tilde{\varphi}(v) = \varphi(v)$ for $v \in E_p^c(\delta_1)$ and $\|\tilde{\varphi}(v) - v\| < \delta$ for $v \in T_p M(\delta_1)$. Put

$$\psi(x) = \begin{cases} \exp_p \circ \tilde{\varphi} \circ \exp_p^{-1}(x) & \text{if } x \in \exp_p(T_p M(\delta_1)), \\ x & \text{otherwise,} \end{cases}$$

and define $\tilde{g} = \psi \circ g$. Let $\hat{\delta} = \hat{\delta}(g, \hat{\varepsilon}) > 0$ be a number as in the definition of persistency of g . Then $d(\tilde{g}, g) < \hat{\delta}$ ($\tilde{g} \in \mathcal{H}(M)$) for a sufficiently small δ . Take and fix $v \in E_p^c(\delta_1)$ such that $d(\exp_p(v), p) = 2\hat{\varepsilon}$. Clearly $g^{mn}(\exp_p(v)) = \exp_p(v)$ for all $m \in \mathbf{Z}$. On the other hand, it is easy to see that for every $y \in B_{\hat{\varepsilon}}(\exp_p(v))$, there is $m(y) \in \mathbf{Z}$ such that

$$d(\tilde{g}^{m(y)n}(y), g^{m(y)n}(\exp_p(v))) = d(\tilde{g}^{m(y)n}(y), \exp_p(v)) > \hat{\varepsilon}.$$

This is a contradiction.

2. PROOF OF PROPOSITION B

Before starting the proof of this proposition, we need some preparation. Throughout this section let $f \in \text{Diff}(M)$ satisfy Axiom A with no-cycles. Take a basic set $\Lambda(f)$ of f and fix $\varepsilon_0 > 0$ sufficiently small. Since $\dim W_{\varepsilon_0}^s(x, f) = \dim W_{\varepsilon_0}^s(y, f)$ for $x, y \in \Lambda(f)$, we denote by $\text{Ind } \Lambda(f)$ the dimension of $W_{\varepsilon_0}^s(x, f)$ for $x \in \Lambda(f)$. If $g \in \text{Diff}(M)$ is C^1 close to f , then the number of basic sets $\{\Lambda_i(g)\}$ of g coincides with that of basic sets $\{\Lambda_i(f)\}$ because of Ω -stability of f .

The following lemma is induced by Franks's lemma (see [9, Lemma 3] for details).

Lemma 1. *Let $\Lambda_1(f)$ and $\Lambda_2(f)$ be basic sets for f . Suppose that there are $p = f^n(p) \in \Lambda_1(f)$ ($n > 0$), $q \in \Lambda_2(f)$ and $x \in M \setminus \Omega(f)$ satisfying $x \in W^s(p, f) \cap W^u(q, f)$. Then, for a neighborhood $\mathcal{U}(f) \subset \text{Diff}(M)$ of f , there are $0 < \varepsilon_1 < \varepsilon_0/2$, $g \in \mathcal{U}(f)$ and two distinct basic sets $\Lambda_1(g)$ and $\Lambda_2(g)$ such that*

$$(i) \quad B_{4\varepsilon_1}(f^i(p)) \cap B_{4\varepsilon_1}(f^j(p)) = \emptyset \text{ for } 0 \leq i \neq j \leq n-1,$$

$$(ii) \quad g(x) = \begin{cases} \exp_{f^{i+1}(p)} \circ D_{f^i(p)} f \circ \exp_{f^i(p)}^{-1}(x) & \text{if } x \in B_{\varepsilon_1}(f^i(p)) \text{ for } 0 \leq i \leq n-1, \\ f(x) & \text{if } x \notin \cup_{i=0}^{n-1} B_{4\varepsilon_1}(f^i(p)), \end{cases}$$

$$(iii) \quad \begin{cases} g^n(p) = p \in \Lambda_1(g) \text{ and } q \in \Lambda_2(g), \\ x \in W^s(p, g) \cap W^u(q, g), \\ T_x W^s(p, g) = T_x W^s(p, f) \text{ and } T_x W^u(q, g) = T_x W^u(q, f). \end{cases}$$

Since f satisfies Axiom A, there exist a Df -invariant continuous splitting $T_{\Omega(f)}M = E^s \oplus E^u$ and a constant $0 < \lambda < 1$ such that

$$\|Df|_{E^s}^m\| < \lambda^m, \|Df|_{E^u}^{-m}\| < \lambda^m$$

for all $m \geq 0$. We denote by E_x^σ a fiber of E^σ at $x \in \Omega(f)$ and put $E_x^\sigma(\varepsilon) = \{v \in E_x^\sigma \mid \|v\| \leq \varepsilon\}$ for $\varepsilon > 0$ ($\sigma = s, u$). Let $g \in \text{Diff}(M)$, $p = g^n(p) \in \Lambda_1(g)$ and $\varepsilon_1 > 0$ be given as in Lemma 1. Then it is easily checked that for $0 < \varepsilon \leq \varepsilon_1$, we have

$$\exp_p(E_p^\sigma(\varepsilon)) \subset W_{\varepsilon_0}^\sigma(p, g) \text{ and } \dim \exp_p(E_p^\sigma(\varepsilon)) = \dim W_{\varepsilon_0}^\sigma(p, g)$$

for $\sigma = s, u$.

Now we shall prove Proposition B. Fix $x \in M \setminus \Omega(f)$ and let $\Lambda_i(f)$ and $\Lambda_j(f)$ be basic sets of f such that $x \in W^s(\Lambda_i(f), f) \cap W^u(\Lambda_j(f), f)$. To simplify the proof we assume $i = 1$ and $j = 2$. If $\text{Ind } \Lambda_1(f) = \dim M$ or $\text{Ind } \Lambda_2(f) = 0$, then the conclusion of this proposition is clear. Thus we shall prove $T_x M = T_x W^s(x, f) + T_x W^u(x, f)$ when $\text{Ind } \Lambda_1(f) \leq \dim M - 1$ and $\text{Ind } \Lambda_2(f) \geq 1$.

Since $f \in \text{int}\mathcal{P}(M)$ and $\Omega(f) = \overline{\mathcal{P}(f)}$, there is $f' \in \text{int}\mathcal{P}(M)$ arbitrarily near to f in the C^1 topology such that

$$(a) \quad f(y) = f'(y) \text{ for all } y \text{ outside of a small neighborhood of } x,$$

$$(b) \quad \text{there are } p = f'^n(p) \in \Lambda_1(f') \text{ for some } n > 0 \text{ and } q \in \Lambda_2(f') \text{ such that } x \in W^s(p, f') \cap W^u(q, f'), T_x W^s(p, f') = T_x W^s(x, f) \text{ and } W^u(q, f') = W^u(x, f).$$

By (a), there are basic sets $\Lambda_i(f')$ ($i = 1, 2$) for f' such that $\Lambda_i(f') = \Lambda_i(f)$ since f is Ω -stable. Let us prove $T_x M = T_x W^s(p, f') + T_x W^u(q, f')$. We identify f' with f for simplicity, and let $\mathcal{U}(f)$ be a small neighborhood of f such that $\mathcal{U}(f) \subset \mathcal{P}(M)$.

Then, by Lemma 1 there are $g \in \mathcal{U}(f)$ and basic sets $\Lambda_i(g)$ ($i = 1, 2$) satisfying Lemma 1 (i), (ii) and (iii). Thus $T_x W^s(p, g) = T_x W^s(x, f)$ and $W^u(q, g) = W^u(x, f)$. Pick $\ell > 0$ such that $g^\ell(x) \in W_{\varepsilon_1/2}^s(p, g)$ and $g^{-\ell}(x) \in W_{\varepsilon_0/2}^u(g^{-\ell}(q), g)$, and define

$$C^u(g^\ell(x)) = \text{the connected component of } g^\ell(x) \text{ in } W^u(g^\ell(q), g) \cap B_{\varepsilon_1}(p).$$

Clearly, $\exp_p^{-1}(C^u(g^\ell(x))) \subset T_p M$.

For a linear subspace E of $T_p M$ and $\nu > 0$, we write

$$E_\nu(g^\ell(x)) = \{v + \exp_p^{-1}(g^\ell(x)) | v \in E \text{ with } \|v\| \leq \nu\}.$$

Then there are a linear subspace $E' \subset T_p M$ and a number $0 < \nu_0 \leq \varepsilon_1$ such that

$$T_{g^\ell(x)} \exp_p(E'_\nu(g^\ell(x))) = T_{g^\ell(x)} C^u(g^\ell(x))$$

and $\exp_p(E'_\nu(g^\ell(x))) \subset B_{\varepsilon_1}(p)$ for $0 < \nu \leq \nu_0$. Since $g^\ell(x) \notin \Omega(g)$, there is $0 < \nu_1 \leq \nu_0$ such that $B_{\nu_1}(g^\ell(x)) \cap g^i(B_{\nu_1}(g^\ell(x))) = \emptyset$ for all $i \in \mathbb{Z} \setminus \{0\}$. If $\mathcal{U}(g) \subset \mathcal{U}(f)$ is a neighborhood of g , then there are $0 < \nu_2 < \nu_1/4$ and $\varphi \in \text{Diff}(M)$ such that

$$\begin{cases} \varphi|_{(B_{4\nu_2}(g^\ell(x)))^c} = id, \\ \varphi(g^\ell(x)) = g^\ell(x), \\ \varphi(\exp_p(E'_{\nu_2}(g^\ell(x)))) \subset C^u(g^\ell(x)), \\ \dim \varphi(\exp_p(E'_{\nu_2}(g^\ell(x)))) = \dim C^u(g^\ell(x)), \\ g' \in \mathcal{U}(g) \text{ where } g' = \varphi^{-1} \circ g. \end{cases}$$

We denote $\exp_p(E'_{\nu_2}(g^\ell(x)))$ by $\exp_p(E'_{\nu_2}(g'^\ell(x)))$ because of $g'^\ell(x) = g^\ell(x)$. It is clear that there are two distinct basic sets $\Lambda_i(g')$ ($i = 1, 2$) such that $\Lambda_i(g') = \Lambda_i(g)$ since g is Ω -stable, and such that

$$\begin{aligned} T_x W^\sigma(x, g') &= T_x W^\sigma(x, g) \quad (\sigma = s, u), \\ \exp_p(E'_{\nu_2}(g'^\ell(x))) &\subset W^u(g'^\ell(x), g') \cap B_{\varepsilon_1}(p), \\ \dim \exp_p(E'_{\nu_2}(g'^\ell(x))) &= \dim W^u(g, g') = \dim C^u(g^\ell(x)). \end{aligned}$$

Lemma 2. *Under the above notation, $\exp_p(E'_{\nu_2}(g'^\ell(x)))$ meets transversely $W_{\varepsilon_1}^s(p, g')$ at $g'^\ell(x)$.*

If this lemma is established, then we have $T_x M = T_x W^s(x, f) + T_x W^u(x, f)$ since $T_x W^\sigma(x, g') = T_x W^\sigma(x, g) = T_x W^\sigma(x, f)$ for $\sigma = s, u$.

Proof of Lemma 2. Put $C_\varepsilon^u(g'^\ell(x)) = B_\varepsilon(g'^\ell(x)) \cap g'^{2\ell}(W_{\varepsilon_0}^u(g'^{-\ell}(q), g'))$ for $\varepsilon > 0$. Take $0 < \tilde{\varepsilon} < \nu_2$ such that $C_{\tilde{\varepsilon}}^u(g'^\ell(x))$ is the connected component of $g'^\ell(x)$ in $B_{\tilde{\varepsilon}}(g'^\ell(x)) \cap g'^{2\ell}(W_{\varepsilon_0}^u(g'^{-\ell}(q), g'))$ for $0 < \varepsilon \leq \tilde{\varepsilon}$, and such that $B_{\tilde{\varepsilon}}(g'^\ell(x)) \cap g'^{2\ell}(W_{\varepsilon_0}^u(g'^{-\ell}(q), g')) \subset \exp_p(E'_{\nu_2}(g'^\ell(x)))$.

Claim. *For every $0 < \varepsilon \leq \tilde{\varepsilon}$, if $d(g'^{-i}(g'^\ell(x)), g'^{-i}(w)) < \varepsilon$ for all $i \geq 0$, then $w \in C_\varepsilon^u(g'^\ell(x))$.*

Indeed, it is clear that $d(g'^{-\ell-i}(x), g'^{-2\ell-i}(w)) < \varepsilon \leq \varepsilon_0/2$ for all $i \geq 0$. On the other hand, since $d(g'^{-\ell-i}(x), g'^{-\ell-i}(q)) < \varepsilon_0/2$ ($\forall i \geq 0$),

$d(g'^{-2\ell-i}(w), g'^{-\ell-i}(q)) \leq d(g'^{-2\ell-i}(w), g'^{-\ell-i}(x)) + d(g'^{-\ell-i}(x), g'^{-\ell-i}(q)) < \varepsilon_0$ for all $i \geq 0$ and so $g'^{-2\ell}(w) \in W_{\varepsilon_0}^u(g'^{-\ell}(q), g')$. Thus $w \in C_\varepsilon^u(g'^\ell(x)) = B_\varepsilon(g'^\ell(x)) \cap g'^{2\ell}(W_{\varepsilon_0}^u(g'^{-\ell}(q), g'))$ since $d(g'^\ell(x), w) < \varepsilon$. The claim is proved.

Suppose that $\exp_p(E'_{\nu_2}(g'^\ell(x)))$ does not meet transversely $W_{\varepsilon_1}^s(p, g')$ at $g'^\ell(x)$. Then there exist $0 < \nu_3 < \min\{\tilde{\varepsilon}, \nu_2/2\}$ such that for every $\delta > 0$ ($\delta \ll \nu_3$) there is $\psi_\delta \in \text{Diff}(M)$ satisfying

$$\begin{cases} \psi_\delta|_{(B_{\nu_3}(g'^\ell(x)))^c} = id, \\ d(\psi_\delta, id) < \delta, \\ \psi_\delta(\exp_p(E'_{\nu_3/2}(g'^\ell(x)))) \cap W_{\varepsilon_1}^s(p, g') = \emptyset. \end{cases}$$

Fix $0 < \varepsilon' < \nu_3/2$ and let $\delta' = \delta'(g', \varepsilon') > 0$ be a number as in the definition of persistency of g' . Take $\delta > 0$ such that $\tilde{g} = g' \circ \psi_\delta \in \mathcal{H}(M)$ and $d(\tilde{g}, g') < \delta'$. Then, for g' -orbit $\{g'^i(x)\}_{i \in \mathbf{Z}}$ of x , there is $y \in B_{\varepsilon'}(g'^\ell(x))$ such that $d(\tilde{g}^i(y), g'^i(g'^\ell(x))) < \varepsilon'$ for all $i \in \mathbf{Z}$. By the claim $y \in \exp_p(E'_{\nu_3/2}(g'^\ell(x)))$, from which $\tilde{g}(y) = g \circ \psi_\delta(y) \notin W_{\varepsilon_1}^s(p, g')$. Thus, by the hyperbolicity, $d(\tilde{g}^i(\tilde{g}(y)), g'^{i+1}(g'^\ell(x))) > \varepsilon'$ for some $i \geq 0$. This is a contradiction.

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