MEAN THEORETIC APPROACH TO THE GRAND FURUTA INEQUALITY

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*Dedicated to Professor Tsuyoshi Ando, the originator of the theory
 of operator means, on his retirement from Hokkaido University*

ABSTRACT. Very recently, Furuta obtained the grand Furuta inequality which is a parametric formula interpolating the Furuta inequality and the Ando-Hiai inequality as follows : If $A \geq B \geq 0$ and A is invertible, then for each $t \in [0, 1]$,

$$F_{p,t}(A, B, r, s) = A^{-r/2} \{A^{r/2} (A^{-t/2} B^p A^{-t/2})^s A^{r/2}\}^{\frac{1-t+r}{(p-t)s+r}} A^{-r/2}$$

is a decreasing function of both r and s for all $r \geq t$, $p \geq 1$ and $s \geq 1$. In this note, we employ a mean theoretic approach to the grand Furuta inequality. Consequently we propose a basic inequality, by which we present a simple proof of the grand Furuta inequality.

1. INTRODUCTION

Throughout this note, we consider bounded linear operators acting on a Hilbert space, simply operators. An operator A is positive if $(Ax, x) \geq 0$ for all $x \in H$. The Löwner-Heinz inequality says that the function $t \rightarrow t^\alpha$ on $[0, \infty)$ is operator monotone for $0 \leq \alpha \leq 1$, i.e.,

$$(1) \quad A \geq B \geq 0 \quad \text{implies} \quad A^\alpha \geq B^\alpha$$

(cf. [13] and [15]). Furuta [7] gave it an ingenious extension which is called the Furuta inequality : If $A \geq B \geq 0$, then

$$(2) \quad (A^r A^p A^r)^{1/q} \geq (A^r B^p A^r)^{1/q}$$

holds for $r \geq 0$, $p \geq 0$ and $q \geq 1$ with $(1 + 2r)q \geq p + 2r$; see [8] for an elementary proof.

Recently Ando and Hiai [2] discussed the log-majorization for positive operators and obtained the following fundamental inequality [2, Theorem 3.5], which is equivalent to their main log-majorization theorem [2, Theorem 2.1]. If $A \geq B \geq 0$ and A is invertible, then the following holds for $p, r \geq 1$:

$$(3) \quad A^r \geq \{A^{r/2} (A^{-1/2} B^p A^{-1/2})^r A^{r/2}\}^{1/p}.$$

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Very recently, Furuta [12] obtained a parameteric formula interpolating the Furuta inequality (2) and the Ando-Hiai inequality (3) in the following manner:

The grand Furuta inequality ([12]). *If $A \geq B \geq 0$ and A is invertible, then for each $t \in [0, 1]$,*

$$(4) \quad F_{p,t}(A, B, r, s) = A^{-r/2} \{A^{r/2} (A^{-t/2} B^p A^{-t/2})^s A^{r/2}\}^{\frac{1-t+r}{(p-t)s+r}} A^{-r/2}$$

is a decreasing function of both r and s for all $r \geq t$, $p \geq 1$ and $s \geq 1$.

In particular, the inequality

$$(5) \quad A^{1-t} = F_{p,t}(A, A, r, s) \geq F_{p,t}(A, B, r, s)$$

holds for $r \geq t$, $p \geq 1$ and $s \geq 1$.

As a matter of fact, (2) and (3) appear in the grand Furuta inequality as the extremal cases $t = 0$ and $t = 1$ with $r = s$ respectively. Therefore we call it the grand Furuta inequality. We note that the original proof in [12] is quite elementary but somewhat technical.

In this note, we employ a mean theoretic approach to the grand Furuta inequality as has been done for the Furuta inequality (see [3], [4], [5], [6], [9], [14], [15]). Thus we propose a basic inequality, by which we present a simple proof of the grand Furuta inequality.

2. MEANS OF OPERATORS

The theory of operator means was established by Kubo and Ando [16], whose heart is the correspondence between operator monotone functions f and means m given by

$$f(t) = 1 \ m \ t \quad (t > 0).$$

In connection with the Löwner-Heinz inequality (1), they exhibit means $\sharp_s$ for $0 \leq s \leq 1$ such that $1 \sharp_s t = t^s$ ($t > 0$), more precisely

$$(6) \quad A \sharp_s B = A^{1/2} (A^{-1/2} B A^{-1/2})^s A^{1/2}$$

for positive invertible operators A and B .

In [10], Furuta proved the monotonicity of the function

$$(7) \quad F(p) = (B^r A^p B^r)^{\frac{1+2r}{p+2r}} \quad \text{for } p \geq 1$$

associated with his inequality under the assumption $A \geq B \geq 0$ and $r \geq 0$. Noting that the monotonicity of $F(p)$ can be rephrased in terms of the function

$$(8) \quad M(p, r) = B^{-2r} \sharp_{\frac{1+2r}{p+2r}} A^p,$$

we showed that $M(p, r)$ is an increasing function of both p and r for all $p \geq 1$ and $r \geq 0$ [4, Theorem 1]; see Lemma 5 below. Moreover, we pointed out that its modification includes Ando's theorem on the geometric mean in [1].

We now rewrite (4) in a mean theoretic form, using the same technique that we used to rewrite (7) as (8). For the sake of convenience, we define $\natural_s$ for $s \in \mathbb{R}$ as in [12] as an extension of $\sharp_s$ for $0 \leq s \leq 1$,

$$(6') \quad A \natural_s B = A^{1/2} (A^{-1/2} B A^{-1/2})^s A^{1/2}$$

for positive invertible operators A and B . For given $A \geq B \geq 0$, $p \geq 1$ and $t \in [0, 1]$, we let

$$(9) \quad F(r, s) = A^{-r+t} \sharp_{\frac{1-t+r}{(p-t)s+r}} (A^t \natural_s B^p)$$

for $r \geq t$ and $s \geq 1$. It is easily checked that $F(r, s) = A^{t/2} F_{p,t}(A, B, r, s) A^{t/2}$ and so the monotonicity of $F_{p,t}(A, B, r, s)$ in (4) is equivalent to that of $F(r, s)$. Similarly, (5) can be rewritten in the form

$$(10) \quad F(r, s) \leq A \quad \text{for } r \geq t \quad \text{and } s \geq 1$$

if $A \geq B \geq 0$ and A is invertible.

Concluding this section, we note that $\natural_s$ is multiplicative in the sense that

$$(11) \quad A \natural_{ab} B = A \natural_a (A \natural_b B)$$

for all a and b .

3. THE GRAND FURUTA INEQUALITY

We begin by stating two simple lemmas by Furuta in [10] and [12]. For the sake of convenience, we give them short proofs.

Lemma 0 ([10]). *For positive operators A and B ,*

$$(ABA)^s = AB^{1/2} (B^{1/2} A^2 B^{1/2})^{s-1} B^{1/2} A$$

holds for $s \geq 1$.

Proof. Let $AB^{\frac{1}{2}} = UH$ be the polar decomposition of $AB^{\frac{1}{2}}$. Then we have for $s \geq 0$

$$\begin{aligned} (ABA)^{1+s} &= (UH^2 U^*)^{1+s} = UH^{2+2s} U^* \\ &= UHH^{2s} HU^* = AB^{1/2} (B^{1/2} A^2 B^{1/2})^s B^{1/2} A. \end{aligned}$$

Lemma 1 ([12]). *If $A \geq B \geq 0$ and A is invertible, then*

$$A^t \natural_s B^p \leq B^{(p-t)s+t}$$

for $p \geq 1$, $1 \leq s \leq 2$ and $0 \leq t \leq 1$.

Proof. It follows from (1) that $A^{-t} \leq B^{-t}$ and moreover

$$\begin{aligned} A^t \natural_s B^p &= A^{t/2} (A^{-t/2} B^p A^{-t/2})^s A^{t/2} \\ &= A^{t/2} A^{-t/2} B^{p/2} (B^{p/2} A^{-t} B^{p/2})^{s-1} B^{p/2} A^{-t/2} A^{t/2} \text{ by Lemma 0} \\ &\leq B^{p/2} (B^{p/2} B^{-t} B^{p/2})^{s-1} B^{p/2} \\ &= B^{(p-t)s+t}. \end{aligned}$$

We use Lemma 1 to prove the following basic inequality which will work well in a proof of the grand Furuta inequality.

Theorem 2. *If $A \geq B \geq 0$ and A is invertible, then*

$$(A^t \natural_s B^p)^{1/(s(p-t)+t)} \leq B$$

for $p \geq 1$, $s \geq 1$ and $0 \leq t \leq 1$.

Proof. It suffices to show that

$$(12) \quad (A^t \natural_{2^k s} B^p)^{1/(2^k s(p-t)+t)} \leq B$$

for $1 \leq s \leq 2$ and $k = 1, 2, \dots$. Lemma 1 says that (12) holds for $k = 0$. So we put $p_1 = s(p-t) + t$, $B_1 = (A^t \natural_s B^p)^{1/p_1}$ and inductively

$$p_{k+1} = 2^k s(p-t) + t \quad \text{and} \quad B_{k+1} = (A^t \natural_{2^k s} B^p)^{1/p_{k+1}}$$

for $k = 1, 2, \dots$. Then we have $p_{k+1} = 2(p_k - t) + t$ and by (11)

$$B_{k+1} = (A^t \natural_2 (A^t \natural_{2^{k-1} s} B^p))^{1/p_{k+1}} = (A^t \natural_2 B_k^{p_k})^{1/p_{k+1}}$$

for $k = 1, 2, \dots$.

It therefore suffices to prove that $B_{k+1} \leq B_k$. For a fixed k , we can assume that $B_k \leq A$ because $B_1 \leq B \leq A$. Hence we apply Lemma 1 to $B_k \leq A$, $p = p_k$ and $s = 2$. It implies that

$$(A^t \natural_2 B_k^{p_k})^{1/(2(p_k-t)+t)} \leq B_k.$$

Finally, since the left-hand side of this is just B_{k+1} by the above remark, we obtain the conclusion $B_{k+1} \leq B_k$.

The second tool is a special case in the Furuta inequality (2).

Lemma 3. *If $A \geq B \geq 0$, then*

$$A \geq (A^{b/2} B^p A^{b/2})^{1/(p+b)}$$

for $p \geq 1$ and $b \geq 0$.

Proof. We take $r = b/2$ and $q = p + b$ in (2). Since p, q and r satisfy the condition which ensures (2), we have the desired inequality.

Based on Theorem 2 and Lemma 3, we give a simple proof of the statement that $F(r, s)$ is a decreasing function of r . The proof we give is similar to that of [4, Theorem 1].

Lemma 4. *If $A \geq B \geq 0$, then $F(r, s)$ is a decreasing function of r for all $r \geq t$.*

Proof. First of all, we put $B_1 = (A^t \natural_s B^p)^{1/((p-t)s+t)}$ as in Theorem 2 and $\langle s \rangle = \frac{1-t+r}{(p-t)s+r}$ for given $s \geq 1$, p , r and t . Since $B_1 \leq B \leq A$ by Theorem 2, Lemma 3 implies that

$$D = (A^{(r-t+d)/2} B_1^{(p-t)s+t} A^{(r-t+d)/2})^{1/((p-t)s+r+d)} \leq A,$$

so that $D^d \leq A^d$ by (1) for $0 < d < 1$. Therefore we have

$$\begin{aligned} F(r, s) &= A^{-r+t} \natural_{\langle s \rangle} B_1^{(p-t)s+t} \\ &= A^{-(r+d-t)/2} (A^d \natural_{\langle s \rangle} A^{(r+d-t)/2} B_1^{(p-t)s+t} A^{(r+d-t)/2}) A^{-(r+d-t)/2} \\ &\geq A^{-(r+d-t)/2} (D^d \natural_{\langle s \rangle} A^{(r+d-t)/2} B_1^{(p-t)s+t} A^{(r+d-t)/2}) A^{-(r+d-t)/2} \\ &= A^{-(r+d-t)/2} (D^d \natural_{\langle s \rangle} D^{(p-t)s+r+d}) A^{-(r+d-t)/2} \\ &= A^{-(r+d-t)/2} D^{1-t+r+d} A^{-(r+d-t)/2} \\ &= A^{-(r+d)+t} \natural_{\frac{1-t+r+d}{(p-t)s+r+d}} B_1^{(p-t)s+t} \\ &= F(r+d, s), \end{aligned}$$

so the proof is complete.

To prove that $F(r, s)$ is a decreasing function of s , we need the following result equivalent to [4, Theorem 3]; cf. [11].

Lemma 5. *If $A \geq B \geq 0$ and $\gamma \geq 0$ is given, then the operator function*

$$f(\alpha, \beta) = A^{-\alpha} \sharp_{\frac{\gamma+\alpha}{\beta+\alpha}} B^{\beta}$$

is a decreasing function of both α and β for all $\alpha \geq 0$ and $\beta \geq \gamma$.

Proof. In [4, Theorem 3], we showed that if $C \geq D \geq 0$ and $\gamma \geq 0$ is given, then

$$g(\alpha, \beta) = D^{-\alpha} \sharp_{\frac{\gamma+\alpha}{\beta+\alpha}} C^{\beta}$$

is an increasing function of both α and β for all $\alpha \geq 0$ and $\beta \geq \gamma$. Since $B^{-1} \geq A^{-1} \geq 0$, it implies that

$$h(\alpha, \beta) = (A^{-1})^{-\alpha} \sharp_{\frac{\gamma+\alpha}{\beta+\alpha}} (B^{-1})^{\beta} = A^{\alpha} \sharp_{\frac{\gamma+\alpha}{\beta+\alpha}} B^{-\beta}$$

is also an increasing function of both α and β for all $\alpha \geq 0$ and $\beta \geq \gamma$. Taking the inverse, we obtain that

$$f(\alpha, \beta) = A^{-\alpha} \sharp_{\frac{\gamma+\alpha}{\beta+\alpha}} B^{\beta} = (A^{\alpha} \sharp_{\frac{\gamma+\alpha}{\beta+\alpha}} B^{-\beta})^{-1} = (h(\alpha, \beta))^{-1}$$

is a decreasing function of both α and β for all $\alpha \geq 0$ and $\beta \geq \gamma$, as required.

Lemma 6. *If $A \geq B \geq 0$, then $F(r, s)$ is a decreasing function of s for all $s \geq 1$.*

Proof. Since $B_1 \leq A$ by Theorem 2, we can apply Lemma 1 to B_1 and A . Namely we have, for $1 \leq s_1 \leq 2$

$$A^t \natural_{s_1} B_1^{p_1} \leq B_1^{(p_1-t)s_1+t}$$

for $p_1 \geq 1$. Taking $p_1 = (p-t)s + t \geq 1$ in particular, we have

$$A^t \natural_{s_1} B_1^{(p-t)s+t} \leq B_1^{(p-t)ss_1+t}.$$

On the other hand, since the left-hand side above is of the form

$$A^t \natural_{s_1} B_1^{(p-t)s+t} = A^t \natural_{s_1} (A^t \natural_s B^p) = A^t \natural_{ss_1} B^p,$$

it follows that

$$A^t \natural_{ss_1} B^p \leq B_1^{(p-t)ss_1+t}.$$

Hence the monotonicity of operator means implies that

$$(13) \quad A^{-r+t} \sharp_{\frac{(p-t)s+r}{(p-t)ss_1+r}} (A^t \natural_{ss_1} B^p) \leq A^{-r+t} \sharp_{\frac{(p-t)s+r}{(p-t)ss_1+r}} B_1^{(p-t)ss_1+t}.$$

Now we apply Lemma 5 to $A \geq B_1$ for $\alpha = r-t, \beta = (p-t)ss_1+t$ and $\gamma = (p-t)s+t$. Then $\gamma \leq \beta$ by $1 \leq s_1$. That is, we have

$$(14) \quad A^{-r+t} \sharp_{\frac{(p-t)s+r}{(p-t)ss_1+r}} B_1^{(p-t)ss_1+t} \leq B_1^{(p-t)s+t} = A^t \natural_s B^p.$$

Combining (13) with (14), we obtain that

$$A^{-r+t} \sharp_{\frac{(p-t)s+r}{(p-t)ss_1+r}} (A^t \natural_{ss_1} B^p) \leq A^t \natural_s B^p.$$

Finally it implies that

$$\begin{aligned}
 F(r, ss_1) &= A^{-r+t} \sharp_{\frac{1-t+r}{(p-t)ss_1+r}} (A^t \natural_{ss_1} B^p) \\
 &= A^{-r+t} \sharp_{\frac{1-t+r}{(p-t)s+r}} (A^{-r+t} \sharp_{\frac{(p-t)s+r}{(p-t)ss_1+r}} (A^t \natural_{ss_1} B^p)) \\
 &\leq A^{-r+t} \sharp_{\frac{1-t+r}{(p-t)s+r}} (A^t \natural_s B^p) \\
 &= F(r, s),
 \end{aligned}$$

so the proof is complete.

Consequently the grand Furuta inequality is proved by Lemmas 4 and 6, in the form of (9).

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