TRANSFORMATIONS CONJUGATE TO THEIR INVERSES HAVE EVEN ESSENTIAL VALUES

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ABSTRACT. Let T be an ergodic automorphism defined on a standard Borel probability space for which T and T^{-1} are isomorphic. We study the structure of the conjugating automorphisms and attempt to gain information about the structure of T . It was shown in *Ergodic transformations conjugate to their inverses by involutions* by Goodson et al. (Ergodic Theory and Dynamical Systems **16** (1996), 97–124) that if T is ergodic having simple spectrum and isomorphic to its inverse, and if S is a conjugation between T and T^{-1} (i.e. S satisfies $TS = ST^{-1}$), then $S^2 = I$, the identity automorphism. We give a new proof of this result which shows even more, namely that for such a conjugation S , the unitary operator induced by T on $L^2(X, \mu)$ must have a multiplicity function whose essential values on the ortho-complement of the subspace $\{f \in L^2(X, \mu) : f(S^2) = f\}$ are always even. In particular, we see that S can be weakly mixing, so the corresponding T must have even maximal spectral multiplicity (regarding ∞ as an even number).

0. INTRODUCTION

Let T be an ergodic transformation defined on a standard Borel probability space $(X, \mathcal{F}, \mu)$ having simple spectrum. It was shown in [2] that if T is isomorphic to its inverse T^{-1} and the conjugating automorphism is S , i.e. $TS = ST^{-1}$, then $S^2 = I$, the identity automorphism. It follows that if T has a conjugating automorphism S for which $S^2 \neq I$, then T has non-simple spectrum. Our aim is to obtain more information about the nature of the spectrum in this situation. In this paper we show that in fact the spectrum of T must have even multiplicity in the orthogonal complement of the subspace $\{f \in L^2(X, \mu) : f(S^2) = f\}$ (we regard ∞ as an even number). Of course this ortho-complement is trivial if T has simple spectrum, and it consists of everything except the constant functions if S is weak mixing. Specifically we prove

Theorem 1. *Suppose that $ST = T^{-1}S$ for some automorphism S ; then the essential values of the multiplicity function of T , restricted to the ortho-complement of the subspace*

$$\{f \in L^2(X, \mu) : f(S^2) = f\},$$

are even.

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This theorem, which is proved in section 3, actually follows from a more general result regarding unitary operators on L^2 -spaces which preserve real-valued functions and are unitarily equivalent to their inverses.

In section 2 we show that if S^2 and T with $ST = T^{-1}S$ are both ergodic, then they both must be weak mixing.

Finally, in the last section we give a number of examples illustrating the types of situations which may arise. In particular, examples for which S is weak mixing are given.

1. PRELIMINARIES

Throughout, $T : (X, \mathcal{F}, \mu) \rightarrow (X, \mathcal{F}, \mu)$ will denote an ergodic automorphism defined on a non-atomic standard Borel probability space. Throughout the identity automorphism will be denoted by I . The group of all automorphisms $\mathcal{U}(X)$ of $(X, \mathcal{F}, \mu)$ becomes a completely metrizable topological group when endowed with the weak convergence of transformations ($T_n \rightarrow T$ if for all $A \in \mathcal{F}$, $\mu(T_n^{-1}(A) \Delta T^{-1}(A)) + \mu(T_n(A) \Delta T(A)) \rightarrow 0$ as $n \rightarrow \infty$). Denote by $C(T)$ the commutant of T , i.e. those automorphisms of $(X, \mathcal{F}, \mu)$ which commute with T . Since we are assuming that the members of $C(T)$ are invertible, $C(T)$ is a group. Suppose that X is an atomic space; then, if $T : X \rightarrow X$ is ergodic, T is a rotation on a finite point set and so is automatically isomorphic to T^{-1} by an involution. Here the theory reduces to the theory of cyclic permutations, and consequently we always assume that X is a non-atomic probability space.

Related to the commutant is the set

$$\mathcal{B}(T) = \{S \in \mathcal{U}(X) : TS = ST^{-1}\}.$$

If S is measure preserving and $TS = ST^{-1}$, then $S^2T = ST^{-1}S = TS^2$. This shows that $\{S^2 : S \in \mathcal{B}(T)\} \subseteq C(T)$.

The spectral properties of T are those of the induced unitary operator defined by

$$U_T : L^2(X, \mu) \rightarrow L^2(X, \mu); \quad U_T f(x) = f(Tx), \quad f \in L^2(X, \mu).$$

Generally a unitary operator $U : H \rightarrow H$ on a separable Hilbert space H is said to have simple spectrum if there exists $h \in H$ such that $Z(h) = H$, where $Z(h)$ is the closed linear span of the vectors $U^n h$, $n \in \mathbb{Z}$. U is determined up to unitary equivalence by a spectral measure class σ and a $\{1, 2, \dots, \infty\}$ -valued multiplicity function M defined on the circle S^1 . The set $\mathcal{M}_U$ of essential spectral multiplicities of U is the set of all σ -essential values of M . The maximal spectral multiplicity (or just multiplicity) of U is $\text{msm}(U) = \sup \mathcal{M}_U$. U has simple spectrum if $\text{msm}(U) = 1$, otherwise U has non-simple spectrum. See Queffelec [7] for a detailed discussion of the spectral properties of dynamical systems.

2. BASIC RESULTS

In this section, we show that if $S \in \mathcal{B}(T)$ where both T and S^2 are ergodic, then they must both be weak mixing. This leads to a simple classification of the members of $\mathcal{B}(T)$. The first part of the next proposition was given in [2]. The second part is well known, consequently we omit their proofs, both being elementary and easy.

Proposition 1. *Let $T : X \rightarrow X$ be an ergodic automorphism.*

- (i) *If $S \in \mathcal{B}(T)$, then $S^{2n} = I$ for some $n > 0$, or S is aperiodic.*

- (ii) If $TS = ST$ where T is weak mixing and S is ergodic, then S is also weak mixing.

Ryzhikov [8] improved on results of Fathi [1] by showing that any automorphism T on a standard Borel probability space $(X, \mathcal{F}, \mu)$ is the product of three involutions (i.e. $T : X \rightarrow X$ can be written $T = U \circ V \circ W$ where $U^2 = V^2 = W^2 = I$). This is best possible (it was mentioned in [2], and it is an elementary fact that T is isomorphic to its inverse via an involution S , if and only if T is the product of two involutions).

In this direction, below we give a simple characterization of transformations which are isomorphic to their inverses.

Proposition 2. *An automorphism $T : X \rightarrow X$ is isomorphic to its inverse if and only if there exist automorphisms S_1 and S_2 for which $T = S_1 \circ S_2^{-1}$ and satisfying $S_1^2 = S_2^2$.*

Proof. First suppose that T is isomorphic to its inverse; then there exists $S_1 \in \mathcal{B}(T)$ satisfying $S_1 T = T^{-1} S_1$. Let $S_2 = T^{-1} \circ S_1$; then $S_2 \in \mathcal{B}(T)$ and

$$S_1 \circ S_2^{-1} = S_1 \circ (T^{-1} \circ S_1)^{-1} = T,$$

and

$$S_2^2 = (T^{-1} \circ S_1)^2 = T^{-1} \circ S_1 \circ T^{-1} \circ S_1 = S_1 \circ T \circ T^{-1} \circ S_1 = S_1^2.$$

Conversely, suppose that $T = S_1 \circ S_2^{-1}$, where $S_1^2 = S_2^2$; then

$$S_1 \circ T = S_1 \circ (S_1 \circ S_2^{-1}) = S_1^2 \circ S_2^{-1} = S_2,$$

and similarly, $T^{-1} \circ S_1 = S_2$, so that $S_1 \in \mathcal{B}(T)$ and T is isomorphic to its inverse. $\square$

The following simple corollary shows that if T is ergodic, the members of $\mathcal{B}(T)$ never commute, and if $S \in \mathcal{B}(T)$, then S^2 has non-simple spectrum. This is mainly of interest when S is also ergodic.

Corollary 1. *If T is ergodic and $S \in \mathcal{B}(T)$, then $C(S^2)$, the commutant of S^2 , is non-abelian. In particular, S^2 has a non-simple spectrum.*

Proof. Suppose there exists $S_1 \in \mathcal{B}(T)$ for which $C(S_1^2)$ is abelian. If we define $S_2 = S_1 \circ T$, we see that $S_1^2 = S_2^2$.

Thus the map $R = S_1^2 = S_2^2$ has 2 distinct square roots S_1, S_2 , which therefore belong to $C(R)$ and hence commute.

Now $T = S_1^{-1} \circ S_2$, so that

$$T^2 = S_1^{-1} \circ S_2 \circ S_1^{-1} \circ S_2 = S_1^{-2} \circ S_2^2 = I,$$

contradicting T being ergodic. $\square$

We now show that if T is ergodic and $S \in \mathcal{B}(T)$ where S^2 is ergodic, then T has to be weak mixing, which in turn implies that S is weak mixing. In a later section we shall give a number of examples of automorphisms conjugated to their inverses by weak mixing transformations.

Theorem 2. *Suppose that $T : X \rightarrow X$ and $S : X \rightarrow X$ are automorphisms for which T and S^2 are ergodic and $ST = T^{-1}S$. Then both S and T are weak mixing.*

Proof. Suppose that $f(Tx) = \lambda f(x)$ for some measurable function f and constant $\lambda \in S^1$. Then

$$f(TSx) = \lambda f(Sx), \text{ so that } f(ST^{-1}x) = \lambda f(Sx).$$

This implies that $\bar{\lambda}f(Sx) = f(STx)$, or that $f \circ S(Tx) = \bar{\lambda}f \circ S(x)$.

Thus $\bar{\lambda}$ is an eigenvalue of T corresponding to the eigenfunction $f \circ S$. In particular

$$\bar{f} \circ S(Tx) = \lambda \bar{f} \circ S(x).$$

The ergodicity of T implies that $\bar{f} \circ S = c \cdot f$ for some constant c , with $|c| = 1$. Then

$$f(S^2x) = f \circ S(Sx) = \bar{c}\bar{f}(Sx) = \bar{c} \cdot cf(x) = f(x).$$

But S^2 is ergodic, so $f = \text{constant}$ and $\lambda = 1$, so that T is weak mixing. Proposition 1 now implies that S^2 and hence S are weak mixing. $\square$

Corollary 2. *Let $S \in \mathcal{B}(T)$ for some ergodic T .*

- (i) *If f is an eigenfunction for T with corresponding eigenvalue λ , then $f \circ S$ is an eigenfunction corresponding to the eigenvalue $\bar{\lambda}$.*
- (ii) *Let H_0 be the closed linear span of the eigenfunctions of T . If for any $S \in \mathcal{B}(T)$, $H_S = \{f \in L^2(X, \mu) : S^2(f) = f\}$, then H_0 and H_S are both T and S invariant subspaces of $L^2(X, \mu)$ which satisfy $H_0 \subseteq H_S$.*

Proof. (i) follows directly from the last theorem.

(ii) can be proved directly, or alternatively we can argue as follows:

Denote by $S_0 = S|_{H_0}$ and $T_0 = T|_{H_0}$. Then $T_0 : H_0 \rightarrow H_0$ has discrete spectrum and hence has simple spectrum. Furthermore $S_0T_0 = T_0^{-1}S_0$ and the conditions of Theorem 1 of [2] are satisfied. (In particular, H_0 may be represented as $L^2(Y)$ for suitable Y corresponding to the maximal Kronecker factor of T .) This implies that $S_0^2 = I$ and the result follows. $\square$

Corollary 2 gives another proof that if S^2 is ergodic, then T is weak mixing (and hence S is weak mixing). This is because the ergodicity of S^2 implies that H_S is the one-dimensional subspace of $L^2(X, \mu)$ consisting only of the constant functions. The same therefore applies to H_0 and the result follows.

The following corollary is now an immediate consequence of Theorem 2 and Proposition 1. The quasi-discrete spectrum transformations of [2] give rise to examples for which (iii) below holds.

Corollary 3. *For each $S \in \mathcal{B}(T)$, one of the following must hold:*

- (i) $S^{2n} = I$ for some $n \geq 1$,
- (ii) S is weakly mixing,
- (iii) S^2 is aperiodic and non-ergodic.

3. CONJUGATION BY A UNITARY OPERATOR

In this section we attempt to explain why all known ergodic automorphisms having a finite Lebesgue component in their spectrum have Lebesgue components of even multiplicity. (See the examples of Mathew-Nadkarni and Lemańczyk, where the Lebesgue component in the spectrum arises in the ortho-complement of the eigenfunctions.) In particular, we show that for a unitary operator $T : L^2(X, \mu) \rightarrow$

$L^2(X, \mu)$, preserving real-valued functions, and $S \in \mathcal{B}(T)$ (also preserving real-valued functions), T has a multiplicity function which only takes even values (a.e.) in the ortho-complement of the subspace $\{f \in L^2(X, \mu) : S^2(f) = f\}$. It follows that if $T : X \rightarrow X$ is an automorphism having simple spectrum and $S \in \mathcal{B}(T)$, then $S^2 = I$, giving a new proof of Theorem 1 of [2].

The following is the main theorem of the paper, and gives Theorem 1 as an immediate consequence. In this case $\mathcal{B}(T)$ denotes the unitary conjugations between T and T^{-1} .

Theorem 3. *Suppose that $T : L^2(X, \mu) \rightarrow L^2(X, \mu)$ is a unitary operator which preserves real-valued functions, and suppose there exists $S \in \mathcal{B}(T)$ which also preserves real-valued functions. Then in the ortho-complement of the subspace*

$$\{f \in L^2(X, \mu) : S^2(f) = f\},$$

the essential values of the multiplicity function of T are even (∞ is considered as an even number).

Proof. Let $H_1 = \{f \in L^2(X, \mu) : S^2(f) = f\}$, $H_{-1} = \{f \in L^2(X, \mu) : S^2(f) = -f\}$ and $H = H_1 \oplus H_{-1}$; then since $ST = T^{-1}S$, H_1 and H_{-1} are both T and S invariant, so the same is true for $H^\perp$.

We shall see that T preserves $\mathcal{P}_1$ and $\mathcal{P}_2$ where

$$(1) \quad H^\perp = \mathcal{P}_1 \oplus \mathcal{P}_2,$$

and where $\mathcal{P}_1$ = those functions whose spectral type is concentrated on the “upper” half, S^+ , of the unit circle in the complex plane (excluding ± 1), and $\mathcal{P}_2$ = those whose spectral type is concentrated on the “lower” half S^- (again excluding ± 1) of the unit circle (spectral type here means with respect to S^2). More precisely, let

$$\mathcal{P}_1 = \{f \in H^\perp : \text{supp } \sigma_f \subseteq S^+\},$$

$$\mathcal{P}_2 = \{f \in H^\perp : \text{supp } \sigma_f \subseteq S^-\}$$

(where $\text{supp } \sigma_f$ denotes the support of the measure σ_f , the spectral type of f with respect to S^2). Since S^2 preserves real-valued functions, it also preserves complex conjugation (i.e. $S^2(\bar{f}) = \overline{S^2(f)}$ for all $f \in L^2(X, \mu)$). It follows that on $H^\perp$, the maximal spectral type of S^2 is invariant under complex conjugation, so if h realizes its maximal spectral type, we can define measures ν_1 and ν_2 on S^1 by

$$\nu_1(A) = \sigma_h(A \cap S^+), \quad \nu_2(A) = \sigma_h(A \cap S^-),$$

and since σ_h is symmetric on S^1 ,

$$\nu_1(\bar{A}) = \sigma_h(\bar{A} \cap S^+) = \sigma_h(\overline{A \cap S^-}) = \sigma_h(A \cap S^-) = \nu_2(A).$$

Now $\nu_i \ll \sigma_h$, $i = 1, 2$, implies that there exists h_1 and h_2 with $\nu_1 = \sigma_{h_1}$, $\nu_2 = \sigma_{h_2}$. It follows that $\nu_2 = \sigma_{\bar{h}_1}$, where

$$h = h_1 + \bar{h}_1,$$

and where $h_1 \in \mathcal{P}_1$, $\bar{h}_1 \in \mathcal{P}_2$. Therefore

$$\mathcal{P}_1 = \{f \in H^\perp : \sigma_f \ll \sigma_{h_1}\},$$

$$\mathcal{P}_2 = \{f \in H^\perp : \sigma_f \ll \sigma_{\bar{h}_1}\}.$$

Here the spectral type is with respect to S^2 , and it is clear that these subspaces are S^2 and T invariant and also that equation (1) holds.

Now we claim that

$$T : \mathcal{P}_1 \rightarrow \mathcal{P}_1 \quad \text{and} \quad T : \mathcal{P}_2 \rightarrow \mathcal{P}_2$$

are unitarily equivalent.

First note that if $f \in \mathcal{P}_1$, then $S(\bar{f}) \in \mathcal{P}_2$ and

$$\begin{aligned} (T^n f, f) &= (ST^n f, Sf) = (T^{-n} Sf, Sf) \\ &= (Sf, T^n Sf) = \overline{(T^n Sf, Sf)} = (T^n S\bar{f}, S\bar{f}). \end{aligned}$$

Consequently, if $Z(f)$ and $Z(S\bar{f})$ are the corresponding cyclic subspaces with respect to T , they are also subspaces of $\mathcal{P}_1$ and $\mathcal{P}_2$ respectively, so that $T|Z(f)$ and $T|Z(S\bar{f})$ are unitarily equivalent (this already shows that T has non-simple spectrum when $H^\perp$ is non-trivial).

Now let

$$\mathcal{P}_1 = \bigoplus_{i=1}^{\infty} Z(k_i)$$

be a spectral decomposition (with respect to T) for $T|_{\mathcal{P}_1}$. Then

$$\sigma_{k_1} \gg \sigma_{k_2} \gg \dots$$

Now $S\bar{k}_i \in \mathcal{P}_2$ and $SZ(\bar{k}_i) = Z(S\bar{k}_i)$, for $i = 1, 2, \dots$. Clearly

$$\bigoplus_{i=1}^{\infty} Z(S\bar{k}_i) = \mathcal{P}_2,$$

with $\sigma_{S\bar{k}_i} = \sigma_{k_i}$, $i = 1, 2, \dots$, and $\sigma_{S\bar{k}_1} \gg \sigma_{S\bar{k}_2} \gg \dots$.

It follows that

$$\bigoplus_{i=1}^{\infty} Z(S\bar{k}_i) \quad \text{is a spectral decomposition for } T|_{\mathcal{P}_2}.$$

The unitary equivalence between $Z(k_i)$ and $Z(S\bar{k}_i)$ for $i = 1, 2, \dots$, now gives the required result.

We now consider the case of $T|_{H_{-1}}$. As before, if $f \in H_{-1}$, then f and $S\bar{f}$ have the same spectral type, but in this case we also have $S\bar{f} \in H_{-1}$. Here we proceed as follows:

$$(T^n f, S\bar{f}) = (ST^n f, S^2 \bar{f}) = (T^{-n} Sf, -\bar{f}) = -(Sf, T^n \bar{f}) = -(T^n f, S\bar{f}).$$

It follows that $(T^n f, S\bar{f}) = 0$ for all $n \in \mathbb{Z}$, so that as before, $T|Z(f)$ and $T|Z(S\bar{f})$ are unitarily equivalent, and similar arguments to those earlier complete the proof. $\square$

4. EXAMPLES

We now give two examples of automorphisms which have conjugations to their inverses which are weak mixing. Suppose that R is a weak mixing transformation having two distinct square roots S_1 and S_2 , i.e. $R = S_1^2 = S_2^2$; then it follows from Proposition 2 that $\hat{T} = S_1 \circ S_2^{-1}$ is isomorphic to its inverse and S_1, S_2 are weak mixing conjugations. It may be that $\hat{T}$ is not ergodic, for example, suppose T is weakly mixing and $R = T^2 \times T^2$, $S_1 = T \times T$ and $S_2(x, y) = (y, T^2 x)$. If

$\widehat{T}(x, y) = S_1 \circ S_2^{-1}(x, y) = (T^{-1}y, Tx)$, then $\widehat{T}^2 = I$. Below we give two examples giving rise to ergodic (and hence weak mixing) $\widehat{T}$.

(i) Let $T : X \rightarrow X$ be a weak mixing automorphism; then $T \times T$ is also weak mixing. Define $S : X \times X \rightarrow X \times X$ by $S(x, y) = (y, Tx)$. Clearly S is a square root of $T \times T$, so is weak mixing. Furthermore, S conjugates $T \times T^{-1}$ to its inverse $T^{-1} \times T$, even if T and T^{-1} are non-isomorphic. This is the situation of the above remark where we have chosen $R = T \times T$, having the two distinct square roots, $S(x, y) = (y, Tx)$ and $S'(x, y) = (Ty, x)$, so that $T \times T^{-1} = S' \circ S^{-1}$.

In [3], del Junco and Lemańczyk have shown that there is a dense G_δ subset $\mathcal{K}$ of $\mathcal{U}(X)$ such that for each $T \in \mathcal{K}$, $C(T \times T^{-1}) = C(T) \times C(T)$. It follows from their results that for $T \in \mathcal{K}$

$$\mathcal{B}(T \times T^{-1}) = \{(P \times Q) \circ F : P, Q \in C(T)\},$$

where F is the flip map, $F(x, y) = (y, x)$. Note that $\mathcal{B}(T \times T^{-1})$ contains many involutions, in fact any map of the form $U(x, y) = (Py, P^{-1}x)$ is such an involution.

Remark. Theorem 3 implies that the essential values of $T \times T^{-1}$ are even for every $T \in \mathcal{U}(X)$, and since it is not hard to see that $T \times T$ is spectrally isomorphic to $T \times T^{-1}$, it follows that the essential values of $T \times T$ are always even. Katok [4] has shown that generically the essential values of $T \times T$ are either $\{2\}$ or $\{2, 4\}$.

(ii) This next example was communicated to us by J. Feldman, and although somewhat similar to example (i), it is of interest because $\mathcal{B}(\widehat{T})$ contains maps of a different nature, namely types of shift maps.

We start with an ergodic automorphism T of $(X, \mathcal{F}, \mu)$, a non-atomic standard Borel space. Let $(Y, \mathcal{B}, m) = \prod_{i=-\infty}^{\infty} (X, \mathcal{F}, \mu)_i$ be the doubly infinite direct product of the space $(X, \mathcal{F}, \mu)$. Define $\widehat{T} : Y \rightarrow Y$ by

$$\widehat{T}(\dots, x_{-1}, x_0^*, x_1, x_2, \dots) = (\dots, T^{-1}x_{-1}, T^*x_0, T^{-1}x_1, Tx_2, \dots),$$

where the $*$ indicates the zeroth coordinate.

Suppose now that $S : Y \rightarrow Y$ is the shift map; then S is an ergodic automorphism and it is easy to check that $S\widehat{T} = \widehat{T}^{-1}S$. In fact S is a mixing automorphism.

Note that there are elements of $\mathcal{B}(\widehat{T})$ which are involutions. For example, define $S_1 : Y \rightarrow Y$ by

$$S_1(\dots, x_{-1}, x_0^*, x_1, x_2, \dots) = (\dots, x_{-1}, x_{-2}, x_1^*, x_0, x_3, x_2, \dots);$$

then $S_1 \in \mathcal{B}(\widehat{T})$ and $S_1^2 = I$.

(iii) Mathew and Nadkarni [6] defined a continuum of examples, each having a spectrum consisting of a discrete component and a Lebesgue component of multiplicity equal to 2. It was shown in [2] that each of these transformations T_ϕ is isomorphic to its inverse, and that every $S \in \mathcal{B}(T_\phi)$ satisfies $S^4 = I$ and $S^2 \neq I$. Consequently the results of sections 2 and 3 are applicable. In fact, using the notation of Theorem 3 we have here the situation where $\mathcal{P}_1 = \{0\} = \mathcal{P}_2$ and $L^2(X, \mu) = H_1 \oplus H_{-1}$, with $H_{-1} \neq \{0\}$.

$T_\phi : [0, 1) \times \mathbb{Z}_2 \rightarrow [0, 1) \times \mathbb{Z}_2$ is defined by $T_\phi(x, g) = (Tx, \phi(x) + g)$ where T is the von Neumann Kakutani adding machine, and ϕ is defined using a “random” Toeplitz sequence with “2 holes”, i.e. at the n th stage in the construction of the 2^n levels of T , ϕ is left undefined on the 2^{n-1} th and top levels of the stack. See [2] for details.

REFERENCES

1. A. Fathi, *Le groupe de transformations de $[0,1]$ qui preservant la mesure de Lebesgue est simple*, Isr. J. Math. **29** (1978), 302-308. MR **58**:6156
2. G.R. Goodson, A. del Junco, M. Lemańczyk, D.J. Rudolph, *Ergodic transformations conjugate to their inverses by involutions*, Ergodic Theory and Dynamical Systems **16** (1996), 97-124.
3. A. del Junco, M. Lemańczyk, *Generic spectral properties of measure preserving maps and actions*, Proc. A.M.S. **115** (1992), 725-736. MR **92i**:28017
4. A. Katok, unpublished lecture notes.
5. M. Lemańczyk, *Toeplitz- $\mathbb{Z}_2$ extensions*, Ann. Inst. Henri Poincaré **24** (1988), 1-43. MR **90b**:28020
6. J. Mathew, M.G. Nadkarni, *A measure preserving transformation whose spectrum has a Lebesgue component of multiplicity two*, Bull. Lond. Math. Soc. **16** (1984), 402-406. MR **86b**:28017
7. M. Queffelec, *Substitution Dynamical Systems, Spectral Analysis*: Lecture Notes in Math., vol. 1294, 1987. MR **89g**:54094
8. V.V. Ryzhikov, *Representation of transformations preserving Lebesgue measure as a composition of periodic transformations*, Mat. Zametky **38** (1985), 860-865; English transl. , Math. Notes **38** (1985), 978-981. MR **87b**:28018

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