OSCILLATORY SINGULAR INTEGRALS ON L^p AND HARDY SPACES

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ABSTRACT. We consider boundedness properties of oscillatory singular integrals on L^p and Hardy spaces. By constructing a phase function, we prove that H^1 boundedness may fail while L^p boundedness holds for all $p \in (1, \infty)$. This shows that the L^p theory and H^1 theory for such operators are fundamentally different.

§1. INTRODUCTION

Let Φ be a real-valued C^1 function on $[-1, 1]$, $\Phi(0) = \Phi'(0) = 0$, $\lambda \in \mathbf{R}$, and define T_λ by

$$(1) \quad T_\lambda f(x) = \text{p.v.} \int_{|x-y| \leq 1} e^{i\lambda\Phi(x-y)} \frac{f(y)}{x-y} dy.$$

Such operators are called oscillatory singular integral operators and have been studied extensively ([2], [3], [6], [9], [10], [12]). If Φ is sufficiently smooth and $\Phi^{(k)}(0) \neq 0$ for some $k > 1$, then the L^p and H^1 boundedness of T_λ is well-known ([13], [6], [7]).

Theorem A. *Suppose Φ is sufficiently smooth and $\Phi^{(k)}(0) \neq 0$ for some $k > 1$; then T_λ are uniformly bounded on $L^p(\mathbf{R})$ ($1 < p < \infty$) and $H^1(\mathbf{R})$.*

The space $H^1(\mathbf{R})$ is the usual Hardy space H^1 .

Both the L^p and H^1 uniform boundedness of $\{T_\lambda\}_{\lambda \in \mathbf{R}}$ may fail if $\Phi^{(n)}(0) = 0$ for all n ([5], [7]). On the other hand, the condition that $\Phi^{(k)}(0) \neq 0$ for some $k > 1$ is not a necessary condition. By using results of Nagel et al. ([4]) on Hilbert transforms along curves, one can obtain the following L^2 result:

Theorem B. *Suppose Φ is even, $\Phi(0) = \Phi'(0) = 0$ and $\Phi'' > 0$. Then the operators T_λ are uniformly bounded on $L^2(\mathbf{R})$ if and only if there is a $C > 0$ such that*

$$(2) \quad \Phi'(Ct) > 2\Phi'(t),$$

for every $t > 0$.

More recent results due to Carlsson et al. ([1]) imply that the uniform L^p boundedness of T_λ holds under exactly the same condition.

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Theorem C. Suppose Φ is given as in Theorem B and $p \in (1, \infty)$. Then T_λ are uniformly bounded on L^p if and only if Φ satisfies (2).

Results concerning odd Φ can be found in [4] and [1]. For the uniform H^1 boundedness of T_λ we have the following ([8]):

Theorem D. Suppose $\Phi(0) = \Phi'(0) = 0$ and

$$(3) \quad \Phi'''(t) \geq 0,$$

for $t > 0$. Then T_λ are uniformly bounded on $H^1(\mathbf{R})$.

To give an example of Φ which has vanishing derivatives at $t = 0$ but satisfies (2) and (3), we cite the function $\Phi(t) = e^{-4/t^2}$. We point out that condition (3) is strictly stronger than condition (2). By using interpolation and Theorem B, it is easy to see that (2) is a necessary condition for the uniform H^1 boundedness of T_λ (when Φ is even and convex). In light of Theorem C, it seems reasonable to speculate that (2) may also be a sufficient condition for the uniform H^1 boundedness of T_λ . This turns out to be false. The purpose of this paper is to construct an even, convex function Φ which satisfies (2) such that the corresponding T_λ 's are not uniformly bounded on $H^1(\mathbf{R})$.

Theorem E. There exists a function Φ which is even and satisfies (i) $\Phi(0) = \Phi'(0) = 0$ and (ii) $\Phi'' > 0$ such that

$$(4) \quad \sup_{\lambda \in \mathbf{R}} \|T_\lambda\|_{L^p \rightarrow L^p} < \infty,$$

for $1 < p < \infty$, and

$$(5) \quad \sup_{\lambda \in \mathbf{R}} \|T_\lambda\|_{H^1 \rightarrow H^1} = \infty.$$

§2. CONSTRUCTION OF Φ

For $k \leq -1$, we choose a function $P_k(t)$ on $[2^{k+1} - 2^k/8, 2^{k+1}]$ such that

$$(i) \quad P_k(2^{k+1} - 2^k/8) = 2^{2k} + 7 \cdot 2^{5k-2}, \quad P_k(2^{k+1}) = 2^{2k+2};$$

$$(ii) \quad P'_k(2^{k+1} - 2^k/8) = 2^{4k+1}, \quad P'_k(2^{k+1}) = 2^{4k+5};$$

(iii) $0 < P'_k(t) < C2^k$, for $t \in [2^{k+1} - 2^k/8, 2^{k+1}]$, where C is some positive constant.

Define g on $[0, 1]$ by $g(0) = 0$ and $g(t) = 2^{2k} + 2^{4k+1}(t - 2^k)$ if $t \in [2^k, 2^{k+1} - 2^k/8]$; $g(t) = P_k(t)$ if $t \in [2^{k+1} - 2^k/8, 2^{k+1}]$. Clearly we have $g \in C^1([0, 1])$. Define $\Phi(t)$ for $t \geq 0$ by

$$\Phi(t) = \int_0^t g(s) ds.$$

For $t \leq 0$ we let $\Phi(t) = \Phi(-t)$. Therefore we have $\Phi \in C^2([-1, 1])$, $\Phi(0) = \Phi'(0) = 0$, and $\Phi''(t) > 0$ for $t \neq 0$. Let T_λ be given by (1). The following result can be found in [14].

Lemma 1. Let $\{n_j\}$ be a sequence of positive integers such that $n_{j+1}/n_j \geq \sigma > 1$ for $j = 1, 2, \dots$, and let $\{\xi_j, \gamma_j\}$ be a sequence of pairs of real numbers. If $\sum_{j=1}^\infty (\xi_j^2 + \eta_j^2) < \infty$, then there is a function $f \in C([0, 2\pi])$ such that

$$\int_0^{2\pi} f(t) \cos n_j t dt = \xi_j,$$

and

$$\int_0^{2\pi} f(t) \sin n_j t dt = \gamma_j,$$

for $j = 1, 2, \dots$.

By Lemma 1, there exists a function $\eta \in C([0, 2\pi])$ such that

$$(6) \quad \int_0^{2\pi} \cos(2^j t) \eta(t) dt = 0;$$

and

$$(7) \quad \int_0^{2\pi} \sin(2^j t) \eta(t) dt = 1/j,$$

for $j = 1, 2, \dots$.

Define $b(t)$ on $\mathbf{R}$ by $b(t) = \eta(t)$ if $t \in [0, 2\pi]$, $b(t) = -\eta(-t)$ if $t \in [-2\pi, 0]$, and $b(t) \equiv 0$ if $t \notin [-2\pi, 2\pi]$. We have $b \in H^1(\mathbf{R})$ and $\|b\|_{H^1} = c_0 > 0$.

Proposition 1. *Let Φ be defined as above. Then for $p \in (1, \infty)$, there exists $C_p > 0$ such that*

$$(8) \quad \sup_{\lambda \in \mathbf{R}} \|T_\lambda\|_{L^p \rightarrow L^p} \leq C_p.$$

Proof. Since $2^{2k} \leq \Phi'(t) \leq 2^{2k+2}$ for $t \in [2^k, 2^{k+1}]$, we have

$$\Phi'(8t) > 2\Phi'(t)$$

for $t \geq 0$. Therefore (8) follows from Theorem C. $\square$

Proposition 2. *Let $a(x)$ be a function defined on $[-\delta, \delta]$ and satisfying $\|a\|_\infty \leq (2\delta)^{-1}$ and*

$$\int_{-\delta}^{\delta} a(x) dx = 0.$$

Then

$$(9) \quad \int_{|x| > 2\delta} \frac{1}{|x|} \left| \int_{-\delta}^{\delta} e^{i\lambda\Phi(x-y)} a(y) dy \right| dx \leq \|T_\lambda a\|_1 + C,$$

for some C which is independent of a and λ .

Proof.

$$\begin{aligned} \|T_\lambda a\|_1 &\geq - \int_{|x| \leq 2\delta} |T_\lambda a(x)| dx \\ &= - \int_{|x| > 2\delta} \int_{-\delta}^{\delta} \left| \frac{1}{x} - \frac{1}{x-y} \right| |a(y)| dy dx + \int_{|x| > 2\delta} \frac{1}{|x|} \left| \int_{-\delta}^{\delta} e^{i\lambda\Phi(x-y)} a(y) dy \right| dx \\ &= -I_1 - I_2 + I_3. \end{aligned}$$

By Proposition 1 and Hölder's inequality, we have

$$I_1 \leq C\delta^{1/2} \|T_\lambda a\|_2 \leq C.$$

For I_2 we have

$$I_2 \leq \int_{2\delta}^{\infty} \frac{\delta^2 \|a\|_{\infty}}{x^2} dx \leq C.$$

Therefore, (9) holds. $\square$

We now prove Theorem E. Let N be a large integer and $\lambda = 2^{N+5}\pi$. Define $a_N(x)$ by

$$a_N(x) = (2^{N/3+4}\pi)b(2^{N/3+4}\pi x).$$

Therefore we have $\text{supp}(a_N) \subset [-2^{-N/3-3}, 2^{-N/3-3}]$, and $\|a_N\|_{H^1} \geq c_0 > 0$ for some c_0 independent of N .

For $k \leq -1$, $t \in [2^k, 2^{k+1} - 2^k/8]$ we have

$$\Phi(t) = \Phi(2^k) + 2^{2k}(t - 2^k) + 2^{4k}(t - 2^k)^2.$$

Therefore, for $-N/3 \leq k \leq -1$,

$$\begin{aligned} & \int_{2^k+2^k/8 \leq x \leq 2^{k+1}-2^k/4} \frac{1}{x} \left| \int_{\mathbf{R}} e^{i\lambda\Phi(x-y)} a_N(y) dy \right| dx \\ & \geq \int_{2^k+2^k/8 \leq x \leq 2^{k+1}-2^k/4} \frac{1}{x} \left| \int_{\mathbf{R}} e^{i\lambda(\Phi(2^k)+2^{2k}(x-y-2^k))} a_N(y) dy \right| dx - C\lambda \cdot 2^{6k} \\ & = \ln(14/9) \left| \int_{\mathbf{R}} e^{-i\lambda \cdot 2^{2k} \cdot y} a_N(y) dy \right| - C\lambda \cdot 2^{6k} \\ & = 2\ln(14/9)(2N/3 + 2k + 1)^{-1} - C \cdot 2^{N+6k}, \end{aligned}$$

where we used (7). Hence we obtain

$$\begin{aligned} & \int_{|x| \geq 2^{-N/3}} \frac{1}{|x|} \left| \int_{\mathbf{R}} e^{i\lambda\Phi(x-y)} a_N(y) dy \right| dx \\ & \geq 2\ln(14/9) \sum_{-N/3 \leq k \leq -N/4} (2N/3 + 2k + 1)^{-1} - C \sum_{k \leq -N/4} 2^{N+6k} \\ & \geq c_1 \ln N, \end{aligned}$$

where c_1 is some absolute constant. By Proposition 2, we have

$$\|T_{\lambda}a_N\|_{H^1} \geq \|T_{\lambda}a_N\|_{L^1} \geq c'_1 \ln N \|a_N\|_{H^1},$$

for $\lambda = 2^{N+5}\pi$. Thus we have

$$\sup_{\lambda \in \mathbf{R}} \|T_{\lambda}\|_{H^1 \rightarrow H^1} = \infty. \quad \square$$

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