OPERATOR VERSIONS OF THE KANTOROVICH INEQUALITY

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ABSTRACT. The Operator Kantorovich Inequality

$$(R^2 - r^2) u^*(a^*a) u \leq R^2 (u^*a^*u)(u^*au)$$

holds for a wide class of operators a on a Hilbert space $\mathcal{H}$ and all operators $u : \mathcal{K} \rightarrow \mathcal{H}$ for which $[a]u$ is a partial isometry, $[a]$ being the range projection of a .

INTRODUCTORY

The Kantorovich Inequality

$$4mM \langle a^{-1}\xi, \xi \rangle \leq (m + M)^2 \frac{\|\xi\|^4}{\langle a\xi, \xi \rangle},$$

valid for a positive definite matrix with least eigenvalue m and largest eigenvalue M , (and any nonzero vector ξ) first appeared in [K] and has been generalized and applied extensively since.

This inequality may be viewed as a ‘reversal’ of the special case

$$\langle a\xi, \xi \rangle \leq \|a\xi\| \|\xi\|$$

of the Cauchy-Schwarz inequality, for it is equivalent to the inequality

$$2\sqrt{mM} \|a\xi\| \|\xi\| \leq (m + M) \langle a\xi, \xi \rangle.$$

The methods of operator and spectral theory allow one to generalize the inequality to a wide class of operators on Hilbert space. If Γ is a nonzero complex number, $R = |\Gamma|$, and $0 \leq r \leq R$, then

$$(R^2 - r^2)^{\frac{1}{2}} \|a\xi\| \|[a]\xi\| \leq R |\langle a\xi, \xi \rangle|, \quad \xi \in \mathcal{H},$$

$[a]$ being the range projection of a , when $a \in \mathcal{C}_{\Gamma, r}$. See below for the definition of $\mathcal{C}_{\Gamma, r}$: it includes all a for which $\sigma(a) \setminus 0$ lies in a disc of radius r with centre Γ , and also the norm-ball of radius r about $\Gamma 1_{\mathcal{H}}$.

This generalizes further:

$$(R^2 - r^2) u^*(a^*a) u \leq R^2 (u^*a^*u)(u^*au)$$

for $a \in \mathcal{C}_{\Gamma, r}$ and all u for which $[a]u$ is a partial isometry.

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NOTATION & TERMINOLOGY

Recall that when $\mathcal{H}$ is a Hilbert space a *projection* on $\mathcal{H}$ is an *hermitian idempotent* in $L(\mathcal{H})$.

A projection e on $\mathcal{H}$ is a *covering projection* for an operator a on $\mathcal{H}$ if $ea = a$. The *range projection* $[a]$ of a is its smallest covering projection.

An operator $a \in L(\mathcal{H})$ is *positive* if $\langle a\xi, \xi \rangle \geq 0$ ($\xi \in \mathcal{H}$): equivalently, if $\sigma(a) \geq 0$, where $\sigma(a)$ is the *spectrum* of a .

Each operator $a \in L(\mathcal{H})$ has a *modulus* $|a| = (a^*a)^{\frac{1}{2}} \geq 0$ in $L(\mathcal{H})$.

Given operators $a \in L(\mathcal{H})$ and $u \in L(\mathcal{K}, \mathcal{H})$ let

$$a^u = u^*au.$$

Then $|a|^{2u} = u^*(a^*a)u$, while $|a^u|^2 = (u^*a^*u)(u^*au)$.

Recall that the operator a is *normal* if $a^*a = aa^*$. If e is a covering projection for a normal operator a , then $ae = a$. (*Proof*: verify that $(ae - a)^*(ae - a) = 0$.)

Operators that are invertible on their range. If the operator a is *invertible on its range*, that is, if the operator $\alpha = [a]a[a] = a[a]$ is invertible in $L([a]\mathcal{H})$, then $m \leq |\alpha| \leq M$ on $[a]\mathcal{H}$, where $M = \|\alpha\|$, $m = \|\alpha^{-1}\|^{-1}$.

A normal operator a is invertible on its range if and only if *either* $0 \in \rho(a)$ *or* $0 \in \sigma(a)$ and 0 is an isolated point of $\sigma(a)$. If, further, $\sigma(a) \setminus 0$ lies in an open half-plane that does not contain 0 , then

$$\sigma(a) \setminus 0 \subseteq \mathcal{D}$$

for some disc $\mathcal{D}$ not containing 0 .

PARTIALLY ISOMETRIC PAIRS

Let $\mathcal{K}$ be a Hilbert space and let $u \in L(\mathcal{K}, \mathcal{H})$. Recall that the operator u is a *partial isometry* if u^*u is a projection: equivalently, if uu^* is a projection.

I shall say that (u, e) is a **partially isometric pair** if eu (or u^*e) is a partial isometry: equivalently, if u^*eu (or euu^*e) is a projection. (This condition has already been considered, though only in the finite-dimensional case: see [BP].)

Three sorts of partially isometric pairs are especially to be noticed: *minimal*, *u-unitary* and *e-identity*.

minimal: For $\xi \in \mathcal{H}$ define

$$u_\xi : \mathbb{C} \rightarrow \mathcal{H} : \lambda \rightarrow \lambda\xi.$$

Then, identifying $\mathbb{C}$ and $L(\mathbb{C})$ canonically,

$$u_\xi^* a u_\xi = \langle a\xi, \xi \rangle, \quad a \in L(\mathcal{H}).$$

So $(u_{\xi/\|e\xi\|}, e)$ is a partially isometric pair provided that $e\xi \neq 0$.

u-unitary: If $u : \mathcal{K} \rightarrow \mathcal{H}$ is unitary then (u, e) is a partially isometric pair for any projection e .

e-identity: If $u : \mathcal{K} \rightarrow \mathcal{H}$ is a partial isometry then $(u, 1_{\mathcal{H}})$ is a partially isometric pair.

Fundamental Lemma. *Let $\mathcal{H}, \mathcal{K}$ be Hilbert spaces, let e be a projection on $\mathcal{H}$ and let $u : \mathcal{K} \rightarrow \mathcal{H}$ be such that (u, e) is a partially isometric pair: i.e. suppose that $e^u (= u^*eu)$ is a projection. Then*

$$eue^u = eu.$$

The *proof* is simple: just multiply out to verify that $(eue^u - eu)^*(eue^u - eu) = 0$.

Corollary. *Let a be an operator on $\mathcal{H}$, let e be a covering projection for a and let (u, e) be a partially isometric pair. Then*

$$e^u a^u = a^u.$$

Further, if a is normal then

$$a^u e^u = a^u.$$

FUNDAMENTAL IDENTITY

Let a be an operator on $\mathcal{H}$, let e be a covering projection for a , and let (u, e) be a partially isometric pair.

Let Γ be any complex number, let $R = |\Gamma|$, and let r be any real number. Then, using the *Corollary to the Fundamental Lemma*,

$$\begin{aligned} |(R^2 - r^2)e^u - \bar{\Gamma}a^u|^2 &= \{(R^2 - r^2)e^u - \Gamma a^{*u}\} \{(R^2 - r^2)e^u - \bar{\Gamma}a^u\} \\ &= (R^2 - r^2)^2 e^{u2} - (R^2 - r^2) \{\bar{\Gamma}e^u a^u + \Gamma a^{*u} e^u\} + R^2 |a^u|^2 \\ &= (R^2 - r^2)^2 e^u - (R^2 - r^2) \{\bar{\Gamma}a^u + \Gamma a^{*u}\} + R^2 |a^u|^2, \end{aligned}$$

while

$$\begin{aligned} \{r^2 e - |a - \Gamma e|^2\}^u &= u^* \{r^2 e - (a^* - \bar{\Gamma}e)(a - \Gamma e)\} u \\ &= -(R^2 - r^2)e^u - |a|^{2u} + \bar{\Gamma}a^u + \Gamma a^{*u} : \end{aligned}$$

so

$$\begin{aligned} R^2 |a^u|^2 - (R^2 - r^2) |a|^{2u} \\ \equiv |(R^2 - r^2)e^u - \bar{\Gamma}a^u|^2 + (R^2 - r^2) \{r^2 e - |a - \Gamma e|^2\}^u, \end{aligned}$$

which is the **Fundamental Identity**.

THE CLASS $\mathcal{C}_{\Gamma, r}$

Let Γ be any nonzero complex number, let $R = |\Gamma|$, and suppose that $0 \leq r \leq R$.

The class $\mathcal{C}_{\Gamma, r}$, defined by

$$\mathcal{C}_{\Gamma, r} = \{a \in L(\mathcal{H}) : |a - \Gamma[a]|^2 \leq r^2[a]\}$$

(where $[a]$ is the range projection of a), is the natural class of operators for which I can prove an *Operator Kantorovich Inequality*.

The class $\mathcal{C}_{\Gamma, r}$ includes

$$\mathcal{N}_{\Gamma, r} = \left\{ a \in L(\mathcal{H}) : a^*a = aa^*, \quad \sigma(a) \setminus 0 \subseteq \mathcal{D}_{\Gamma, r} \right\},$$

where

$$\mathcal{D}_{\Gamma, r} = \{z \in \mathbb{C} : |z - \Gamma| \leq r\},$$

and also the norm-ball

$$\mathcal{B}_{\Gamma, r} = \{a \in L(\mathcal{H}) : \|a - \Gamma 1_{\mathcal{H}}\| \leq r\}.$$

Remark. If a is normal and $a \in \mathcal{C}_{\Gamma, r}$ then $|a| \in \mathcal{C}_{|\Gamma|, r}$.

Remark. Every normal operator a for which $\sigma(a) \setminus 0$ lies in an open half-plane (not containing 0) belongs to $\mathcal{N}_{\Gamma, r}$ for some Γ and r .

Remark. The operator with matrix $\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$ does not belong to $\mathcal{C}_{\Gamma, r}$ for any Γ and r .

Positive Operators in $\mathcal{N}_{\Gamma, r}$. Suppose that $a \geq 0$ and that a is invertible when restricted to its range: either $0 \in \rho(a)$ or if $0 \in \sigma(a)$ then 0 is an isolated point of $\sigma(a)$. Let

$$m = \min\{\sigma(a) \setminus 0\}, \quad M = \max\sigma(a) = \|a\|.$$

Then $a \in \mathcal{N}_{\Gamma, r}$ with $\Gamma = \frac{m+M}{2}$, $r = \frac{M-m}{2}$.

THE INEQUALITIES

Let Γ be any nonzero complex number, let $R = |\Gamma|$, and let $0 \leq r \leq R$.

Theorem 1 (Operator Kantorovich Inequality). *Let $\mathcal{H}$ and $\mathcal{K}$ be Hilbert spaces. Suppose that $a \in \mathcal{C}_{\Gamma, r}$ and let $[a]$ be the range projection of a . Let $u \in L(\mathcal{K}, \mathcal{H})$ be such that $(u, [a])$ is a partially isometric pair. Then*

$$(R^2 - r^2) |a|^{2u} \leq R^2 |a^u|^2.$$

Proof. Immediate from the Fundamental Identity.

Corollary 1. *Let a be a positive operator on $\mathcal{H}$ which is invertible on its range, let $m = \min\{\sigma(a) \setminus 0\}$, $M = \max\sigma(a) = \|a\|$. Let $u \in L(\mathcal{K}, \mathcal{H})$ be such that $(u, [a])$ is a partially isometric pair. Then*

$$2\sqrt{mM} (a^{2u})^{\frac{1}{2}} \leq (m + M) a^u.$$

This follows from Theorem 1 and from the uniqueness and monotonicity of square roots.

Corollary 2 (Scalar Version — Minimal Partially Isometric Pairs). *Let $a \in \mathcal{C}_{\Gamma, r}$, and let $[a]$ be its range projection. Then*

$$(R^2 - r^2)^{\frac{1}{2}} \|a\xi\| \|[a]\xi\| \leq R |\langle a\xi, \xi \rangle|, \quad \xi \in \mathcal{H}.$$

If a is positive with $\sigma(a) \setminus 0 \subseteq [m, M]$ ($0 < m \leq M$), then

$$2\sqrt{mM} \|a\xi\| \|[a]\xi\| \leq (m + M) \langle a\xi, \xi \rangle, \quad \xi \in \mathcal{H}.$$

Proof. There is nothing to prove if $[a]\xi = 0$: otherwise put $u = u_{\xi/\|[a]\xi\|}$ and take square roots. The second assertion is a direct consequence of the first.

Remark. The second assertion may be proved ‘in one line’:

$$\begin{aligned} & (m + M)^2 \langle a\xi, \xi \rangle^2 - 4mM \|a\xi\|^2 \|[a]\xi\|^2 \\ &= \left\{ 2mM \|[a]\xi\|^2 - (m + M) \langle a\xi, \xi \rangle \right\}^2 \\ &\quad + 4mM \left\langle (M - a)(a - m)[a]\xi, [a]\xi \right\rangle \|[a]\xi\|^2 \\ &\geq 0. \end{aligned}$$

Remark. The example $a = \text{diag}\{1, -1\}$ shows that the hypothesis on the spectrum is essential. The example $a = \text{diag}\{m, M\}$, $\xi = (M^{\frac{1}{2}}, m^{\frac{1}{2}})^*$ shows that the ratio $\frac{m+M}{2\sqrt{mM}}$ may be attained so cannot be bettered.

Corollary 2 can be ‘polarized’:

Corollary 3. *Let $a \in \mathcal{C}_{\Gamma, r}$ and let $[a]$ be its range projection. Then*

$$(R^2 - r^2) |\langle a\xi, a\eta \rangle \langle [a]\xi, [a]\eta \rangle| \leq R^2 |\langle a\xi, \xi \rangle \langle a\eta, \eta \rangle|, \quad \xi, \eta \in \mathcal{H}.$$

The proof:

$$\begin{aligned} (R^2 - r^2) |\langle a\xi, a\eta \rangle \langle [a]\xi, [a]\eta \rangle| &\leq (R^2 - r^2) \|a\xi\| \|a\eta\| \|[a]\xi\| \|[a]\eta\| \\ &= (R^2 - r^2)^{\frac{1}{2}} \|a\xi\| \|[a]\xi\| (R^2 - r^2)^{\frac{1}{2}} \|a\eta\| \|[a]\eta\| \end{aligned}$$

which, by Corollary 2,

$$\begin{aligned} &\leq R |\langle a\xi, \xi \rangle| R |\langle a\eta, \eta \rangle| \\ &= R^2 |\langle a\xi, \xi \rangle \langle a\eta, \eta \rangle|. \end{aligned}$$

The next corollary provides an improvement of both the result and the proof of Strang [S].

Corollary 4. *Suppose that a is an operator on the Hilbert space $\mathcal{H}$ and that a is invertible on its range: i.e. the operator $\alpha = eae$ is invertible in $L(e\mathcal{H})$, where e is the range projection of a . Let $M = \|\alpha\|$, $m = \|\alpha^{-1}\|^{-1}$. Then*

$$\begin{aligned} 4mM |\langle ae\xi, a\eta \rangle \langle e\xi, e\eta \rangle| &\leq (m + M)^2 \langle |\alpha|\xi, \xi \rangle \langle |\alpha|\eta, \eta \rangle, \\ 4mM |\langle ae\xi, e\eta \rangle \langle e\xi, a^{-1}e\eta \rangle| &\leq (m + M)^2 \|e\xi\|^2 \|e\eta\|^2, \quad \xi, \eta \in \mathcal{H}. \end{aligned}$$

Proof. I abuse notation conventionally and write $a^{-1}e\eta$ in place of $\alpha^{-1}e\eta$.

As remarked above, $m \leq |\alpha| \leq M$ on $e\mathcal{H}$. Corollary 3 applied to the operator α and vectors $e\xi, e\eta$ gives the first conclusion.

Now given $\xi, \eta \in \mathcal{H}$ there are (unique) $\theta, \phi \in e\mathcal{H}$ such that $e\xi = |\alpha|^{\frac{1}{2}}\theta$, $e\eta = \alpha|\alpha|^{-\frac{1}{2}}\phi$. Then

$$\begin{aligned} \langle ae\xi, e\eta \rangle &= \langle |\alpha|\theta, |\alpha|\phi \rangle, \\ \langle e\xi, a^{-1}e\eta \rangle &= \langle \theta, \phi \rangle, \\ \|e\xi\|^2 &= \langle |\alpha|\theta, \theta \rangle, \\ \|e\eta\|^2 &= \langle |\alpha|\phi, \phi \rangle. \end{aligned}$$

Thus

$$4mM |\langle ae\xi, \eta \rangle \langle e\xi, a^{-1}e\eta \rangle| = 4mM |\langle |\alpha|\theta, |\alpha|\phi \rangle \langle \theta, \phi \rangle|$$

which, by the first part of this corollary,

$$\begin{aligned} &\leq (m + M)^2 \langle |\alpha|\theta, \theta \rangle \langle |\alpha|\phi, \phi \rangle \\ &= (m + M)^2 \|e\xi\|^2 \|e\eta\|^2. \end{aligned}$$

This immediately specialises to yield

Corollary 5 (Original Kantorovich Inequality). *If a is positive and invertible, with $\sigma(a) \subseteq [m, M]$ ($0 < m \leq M$), then*

$$4mM \langle a^{-1}\xi, \xi \rangle \leq (m + M)^2 \frac{\|\xi\|^4}{\langle a\xi, \xi \rangle}, \quad \xi \in \mathcal{H}.$$

Remark. This may also be shown ‘in one line’:

$$\begin{aligned} & (m+M)^2 \|\xi\|^4 - 4mM \langle a\xi, \xi \rangle \langle a^{-1}\xi, \xi \rangle \\ &= \left\{ 2mM \langle a^{-1}\xi, \xi \rangle - (m+M) \|\xi\|^2 \right\}^2 \\ &\quad + 4mM \left\langle (M-a)(1-ma^{-1})\xi, \xi \right\rangle \langle a^{-1}\xi, \xi \rangle \\ &\geq 0. \end{aligned}$$

‘DUPLEX’ INEQUALITIES

Suppose that a and b are operators on $\mathcal{H}$ and that e is a common covering projection for a and b : i.e. $ea = a$, $eb = b$. Suppose further that (u, e) is a partially isometric pair.

Let Γ be any complex number, let $R = |\Gamma|$, and let r be any real number. Then

$$\begin{aligned} & |2(R^2 - r^2)e^u - \bar{\Gamma}(a+b)^u|^2 \\ &= \{2(R^2 - r^2)e^u - \Gamma(a+b)^{*u}\} \{2(R^2 - r^2)e^u - \bar{\Gamma}(a+b)^u\} \\ &= 4(R^2 - r^2)^2 e^u + R^2 |(a+b)^u|^2 - 2(R^2 - r^2) \{ \bar{\Gamma}(a+b)^u + \Gamma(a+b)^{*u} \} \end{aligned}$$

while

$$\begin{aligned} & \{|a-b|^2 + 2r^2e - |a-\Gamma e|^2 - |b-\Gamma e|^2\}^u \\ &= -(a^*b + b^*a)^u - 2(R^2 - r^2)e^u + \bar{\Gamma}(a+b)^u + \Gamma(a+b)^{*u} : \end{aligned}$$

so

$$\begin{aligned} & R^2 |(a+b)^u|^2 - 2(R^2 - r^2)(a^*b + b^*a)^u \\ &\equiv |2(R^2 - r^2)e^u - \bar{\Gamma}(a+b)^u|^2 \\ &\quad + 2(R^2 - r^2) \{|a-b|^2 + 2r^2e - |a-\Gamma e|^2 - |b-\Gamma e|^2\}^u, \end{aligned}$$

which is the **Duplex Fundamental Identity**. The next theorem follows from this.

Theorem 2 (Duplex Kantorovich Inequality). *Suppose that a and b are operators on $\mathcal{H}$ with common range projection e and that both a and b belong to the class $\mathcal{C}_{\Gamma, r}$. Suppose also that (u, e) is a partially isometric pair. Then*

$$2(R^2 - r^2)(a^*b + b^*a)^u \leq R^2 |(a+b)^u|^2.$$

When $a = b$ one recovers

$$(R^2 - r^2) |a|^{2u} \leq R^2 |a^u|^2.$$

Theorem 3 (Positive Duplex Case). *Suppose that a and b are both positive operators on $\mathcal{H}$, that they commute, have common range projection e , and are invertible on their range. Let*

$$\begin{aligned} m_1 &= \min\{\sigma(a) \setminus 0\}, & M_1 &= \max\sigma(a), \\ m_2 &= \min\{\sigma(b) \setminus 0\}, & M_2 &= \max\sigma(b). \end{aligned}$$

Then

$$4(m_1M_2 + m_2M_1)(ab)^u \leq |(m_2a + M_1b)^u|^2 + |(M_2a + m_1b)^u|^2$$

for

$$\begin{aligned} & |(m_2a + M_1b)^u|^2 + |(M_2a + m_1b)^u|^2 - 4(m_1M_2 + m_2M_1)(ab)^u \\ &= |2m_2M_1e^u - (m_2a + M_1b)^u|^2 + |2m_1M_2e^u - (M_2a + m_1b)^u|^2 \\ &+ 4m_2M_1\{e(M_1 - a)(b - m_2)e\}^u + 4m_1M_2\{e(M_2 - b)(a - m_1)e\}^u. \end{aligned}$$

This may be compared with the *Generalized Pólya-Szegő Inequality* of Greub & Rheinboldt [GR], which the next theorem extends.

Theorem 4. *Suppose that a and b are commuting normal operators, with common range projection e , and are invertible on their range. Let*

$$\begin{aligned} m_1 &= \|(eae)^{-1}\|^{-1}, & M_1 &= \|eae\|, \\ m_2 &= \|(ebe)^{-1}\|^{-1}, & M_2 &= \|ebe\|. \end{aligned}$$

Then

$$\begin{aligned} & 4m_1m_2M_1M_2 \left| \langle a\xi, a\eta \rangle \langle b\xi, b\eta \rangle \right| \\ & \leq (m_1m_2 + M_1M_2)^2 \langle |a|\xi, |b|\xi \rangle \langle |a|\eta, |b|\eta \rangle, \quad \xi, \eta \in \mathcal{H}. \end{aligned}$$

If $(eae)(ebe)^{-1} \geq 0$, if, in particular, $a \geq 0$, $b \geq 0$, then

$$\begin{aligned} & 4m_1m_2M_1M_2 \left| \langle a\xi, a\eta \rangle \langle b\xi, b\eta \rangle \right| \\ & \leq (m_1m_2 + M_1M_2)^2 \langle a\xi, b\xi \rangle \langle a\eta, b\eta \rangle, \quad \xi, \eta \in \mathcal{H}. \end{aligned}$$

Proof. Note that $\|(eae)(ebe)^{-1}\| \leq \frac{M_1}{m_2}$, while $\|(ebe)(eae)^{-1}\|^{-1} \geq \frac{m_1}{M_2}$; apply Corollary 4 to the operator $(eae)(ebe)^{-1}$ and the vectors $b\xi$ and $b\eta$.

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