A NOTE ON MILLER'S THEOREM ABOUT MAPS OUT OF CLASSIFYING SPACES

C. A. MCGIBBON

(Communicated by Thomas Goodwillie)

ABSTRACT. Let X be a connected infinite loop space whose fundamental group is a torsion group and let Y be a finite nilpotent CW -complex. The main result of this paper is that the space of based maps from X to the profinite completion of Y is weakly contractible.

1. INTRODUCTION

The theorem to which the title refers is Haynes Miller's celebrated result,

Theorem 1. *Let G be a discrete, locally finite group and let Y be a connected finite dimensional CW -complex. Then the space of based maps from the classifying space of G to Y is weakly contractible.*

Given two spaces X and Y with base points, let $Map_*(X, Y)$ denote the function space of all maps from X to Y which preserve those base points. The iterated loop space $\Omega^k Map_*(X, Y)$ is easily seen to be homeomorphic to $Map_*(X, \Omega^k Y)$. Thus if X is arcwise connected, the weak contractibility of $Map_*(X, Y)$ is equivalent to the condition that every based map from X to $\Omega^k Y$ is null homotopic for every $k \geq 0$.

Miller's theorem is the crucial ingredient in the following result. In it $\widehat{Y}$ denotes the profinite completion of a space Y in the sense of Sullivan [1], [8]. The first case in this theorem is essentially due to Zabrodsky [9], while the second is due to Friedlander and Mislin [3].

Theorem 2. *$Map_*(X, \widehat{Y})$ is weakly contractible if Y is a nilpotent finite CW -complex and if either*

- 1) *X is a connected CW -complex whose fundamental group is locally finite and $\pi_n X = 0$ for n sufficiently large, or*
- 2) *$X = BG$ where G is a Lie group with only a finite number of path components.*

This result provides one with a large source of *phantom maps* ([5], Section 5). This term will be used in this paper to describe any based map $f : X \rightarrow Y$ with the property that for any finite CW -complex K and any map $g : K \rightarrow X$, the

Received by the editors January 10, 1995 and, in revised form, April 4, 1995.
 1991 *Mathematics Subject Classification.* Primary 55R35, 55P47, 55S37.

composition fg is null homotopic.¹ When Y is a nilpotent target of finite type, the profinite completion $Y \rightarrow \hat{Y}$ induces a function $Map_*(X, Y) \rightarrow Map_*(X, \hat{Y})$ whose kernel is precisely the set of all phantom maps from X to Y . Thus Theorem 2 asserts that every map from X to $\Omega^k Y$ is a phantom map when X and Y satisfy the stated conditions.

The purpose of this note is to derive another result similar to Theorem 2 from Miller's Theorem.

Theorem 3. *Let X be a connected infinite loop space whose fundamental group is a torsion group and assume that Y is a nilpotent finite CW-complex. Then the space of based maps from X to $\hat{Y}$ is weakly contractible.*

Remarks. 1) Notice that it is necessary in Theorem 3 that $\pi_1 X$ does not contain an infinite cyclic summand. Otherwise X would dominate S^1 and thus $Map_*(X, \hat{Y})$ would be weakly contractible if and only if $\Omega \hat{Y}$ was.

2) The weak contractibility of $Map_*(X, \hat{Y})$ in Theorems 2 and 3 implies that, for all integers $k \geq 0$,

$$[X, \Omega^k Y] \approx [X_o, \Omega^k Y]$$

by Theorem 5.4 of [5]. Here X_o denotes the rationalization of X and $[X, Z]$ denotes the set $\pi_0 Map_*(X, Z)$. It then follows from Theorem 5.2, *ibid.*, that

$$[X_o, Z] \approx \prod_{j \geq 1} H^j(X; \pi_{j+1}(Z) \otimes \mathbb{R}),$$

where $\mathbb{R}$ is a rational vector space whose cardinality equals that of the real numbers. Moreover, when Z is rationally an H -space, there is a natural group structure on $[X_o, Y]$ so that the above bijection is an isomorphism of rational vector spaces. Thus in these special cases one has a simple and complete classification of homotopy classes of maps (all of which are phantoms) from X to the iterated loop spaces of Y .

Proof of Theorem 3. We start with a special case, where X is a path component of QS^0 . As usual, $QA = \text{colim } \Omega^n \Sigma^n A$ for a space A . Let $\mathcal{S}_\infty$ denote the infinite symmetric group. There is a map

$$B\mathcal{S}_\infty \longrightarrow X$$

which induces an isomorphism in integral homology and in mod p homology for all primes p [7]. Let C denote the cofiber of this map and note that C is acyclic. The cofiber sequence

$$B\mathcal{S}_\infty \longrightarrow X \longrightarrow C$$

induces a fibration

$$Map_*(B\mathcal{S}_\infty, Y) \longleftarrow Map_*(X, Y) \longleftarrow Map_*(C, Y),$$

for any based target Y . Assume now that Y is finite dimensional. The group $\mathcal{S}_\infty$ is locally finite as it is isomorphic to the direct limit of the finite symmetric groups. Thus the base space of this fibration is weakly contractible by Miller's Theorem.

¹These are referred to as phantom maps of the second kind in [5]. Phantom maps of the first kind are maps out of a CW-complex whose restrictions to each skeleton are null-homotopic. Of course when the domain is a CW-complex with only a finite number of cells in each dimension, the two kinds of phantom maps coincide.

Assume in addition that Y is nilpotent and let $Y^{(n)}$ denote the n th stage in the principal refinement of its Postnikov decomposition. It follows that $[C, Y^{(n)}] = *$ for each n , by an easy induction argument. Thus the only maps from C to Y are phantom maps. However, if Y has finite type, it follows from Example 4.1 of [4] that there are no essential phantom maps from C to Y . Thus if Y is a nilpotent finite complex, it follows that $\text{Map}_*(C, Y)$ is weakly contractible and hence the same is true of $\text{Map}_*(X, Y)$. $\square$

Now let $n \geq 2$ and consider the principal fibration sequence

$$F_n \longrightarrow QS^n \longrightarrow K(\mathbb{Z}, n)$$

where the second map is an n -equivalence. To this fibration we will apply the following result from [6]. It is often referred to as the Zabrodsky lemma.

Lemma 4. *Let $E \rightarrow B$ be a principal bundle with structure group G . If $\text{Map}_*(G, Y)$ is contractible, then $\text{Map}_*(B, Y) \rightarrow \text{Map}_*(E, Y)$ is a homotopy equivalence.*

Choose a simplicial model $G \rightarrow E \rightarrow B$ for the fibration $F_n \rightarrow QS^n \rightarrow K(\mathbb{Z}, n)$. Let Y be a nilpotent finite complex and assume for the moment that $\text{Map}_*(G, \hat{Y})$ is contractible. The contractibility of $\text{Map}_*(B, \hat{Y})$ follows at once from the first part of Theorem 2. The weak contractibility of $\text{Map}_*(QS^n, \hat{Y})$ is then a consequence of the Zabrodsky lemma. To show that $\text{Map}_*(G, \hat{Y})$ is contractible note that $F_{n-k} \simeq \Omega^k F_n$ and note that the case of F_0 was established at the beginning of the proof. The weak contractibility of $\text{Map}_*(F_n, \hat{Y})$ then follows by n applications of the Zabrodsky lemma to fibrations of the form

$$F_{k-1} \longrightarrow * \longrightarrow F_k.$$

Suppose that K and L are finite CW -complexes which occur in a cofiber sequence

$$S^n \longrightarrow K \longrightarrow L.$$

The functor $Q(\)$ converts this into a principal fibration sequence

$$QS^n \longrightarrow QK \longrightarrow QL.$$

Hence if $n \geq 2$ and $\text{Map}_*(QK, \hat{Y})$ is weakly contractible, the Zabrodsky lemma implies the same is true of $\text{Map}_*(QL, \hat{Y})$. Thus, by induction, Theorem 3 is true whenever the source is an infinite loop space of the form QW where W is a 1-connected finite complex.

Let E be a 1-connected infinite loop space and let $f : E \rightarrow \Omega^k \hat{Y}$ be any map. Let $g : K \rightarrow E$ be a map of a finite CW -complex into E . Since E is 1-connected, the map g factors through a 1-connected finite CW -complex, say $W = K/K_1$. The quotient map $g' : W \rightarrow E$ then factors through QW as in the following diagram:

$$\begin{array}{ccccc} & & QW & & \\ & \nearrow & \uparrow & \searrow & \\ & & W & \xrightarrow{g'} & E \\ & \nearrow & \uparrow & \searrow & \\ & & K & \xrightarrow{g} & E \end{array} \quad \xrightarrow{f} \quad \Omega^k \hat{Y}.$$

Since we already know that Theorem 3 holds for QW , it follows that the composition fg is null-homotopic and thus f is a phantom map. Since there are no essential phantom maps into a complete space, it follows that Theorem 3 now holds for all

1-connected infinite loop spaces. The non-simply-connected case follows using the fibration

$$\tilde{E} \longrightarrow E \longrightarrow K(\pi_1 E, 1),$$

where the second map induces an isomorphism on fundamental groups. The Zabrodsky lemma together with Miller's theorem applied to the base implies that $\text{Map}_*(E, \hat{Y})$ is weakly contractible and completes the proof of Theorem 3.

I would like to acknowledge the referee's valuable help on this paper. Among other things, he pointed out that Zabrodsky's Theorem 2.1 should not require the domain X to be simply connected or to have finite type as it originally did. The referee's proof goes as follows. In the simply connected version one can use Lemma 4 to reduce the theorem to the case $X = K(\pi, n)$ or even to the case $X = K(\pi, 2)$. Choose a free resolution of π

$$0 \longrightarrow F_0 \longrightarrow F_1 \longrightarrow \pi \longrightarrow 0,$$

and consider the associated fibration sequence

$$K(F_0, 2) \longrightarrow K(F_1, 2) \longrightarrow K(\pi, 2).$$

Another application of Lemma 4 shows that it is enough to prove Theorem 2 in the case $X = K(F, 2)$ for a free abelian group F . Now consider the fibration sequence

$$K(\mathbb{Q}/\mathbb{Z} \otimes F, 1) \longrightarrow K(F, 2) \longrightarrow K(\mathbb{Q} \otimes F, 2).$$

The theorem is clearly true in the case $X = K(\mathbb{Q} \otimes F, 2)$, since this space is rational, so it is enough (again by Lemma 4) to prove it for $X = K(\mathbb{Q}/\mathbb{Z} \otimes F, 1)$. This then follows directly from Miller's Theorem and an arithmetic square argument ([6], Theorem 1.5).

To finish the proof of Theorem 2.1 in the non-simply-connected case one has to deal with the possibility that X might not be a simple space. In this case one uses the following unbased enhancement of the Zabrodsky lemma.

Lemma 4'. *Let $F \rightarrow E \rightarrow B$ be a bundle where B is a connected CW-complex. If $F \rightarrow *$ induces weak equivalence $Y \rightarrow \text{Map}(F, Y)$, then $E \rightarrow B$ induces a weak equivalence $\text{Map}(B, Y) \rightarrow \text{Map}(E, Y)$.*

This is easily proved by induction on the cells of B .

The referee also pointed out the following very different proof of Theorem 3. As has been shown already it suffices to prove the result for QW where W is a finite 1-connected complex. After replacing W if necessary by the realization of its singular complex, we can assume that $W = |K|$ for some simplicial set K . Consider the n -fold product W^n as a G -space where G is the symmetric group on n letters. It is clear that W^n can be given the structure of a G -CW-complex. ($W^n = |K^n|$ in such a way that the action of G on W^n is the geometric realization of an action of G on K^n ; the cells of W^n then correspond to the nondegenerate simplices of K^n , and they are permuted in a G -CW way by the action of G .)

Consider the quotient map

$$(W^n \times EG)/G \rightarrow W^n/G = \text{SP}^n W$$

and observe that W^n/G inherits a CW-structure from the G -CW-structure on W^n . Let e be the interior of a cell in W^n/G . It is clear that the inverse image of e in $(W^n \times EG)/G$ is $e \times BH$, where H is the isotropy subgroup of e . One then

sees by induction on the cell decomposition of W^n/G that the above quotient map induces a weak equivalence

$$\mathrm{Map}(\mathrm{SP}^n W, Y) \rightarrow \mathrm{Map}((W^n \times EG)/G, Y)$$

for a finite dimensional target Y . Passing to the limit and identifying QW as a completion of $\mathrm{colim}(W^n \times E\Sigma_n)/\Sigma_n$ (see [2], pp. 50–59), one concludes that the map

$$QW \rightarrow \mathrm{SP}^\infty W$$

induces a weak equivalence

$$\mathrm{Map}(\mathrm{SP}^\infty W, Y) \rightarrow \mathrm{Map}(QW, Y).$$

The proof is completed by noting that $\mathrm{SP}^\infty W$ has the homotopy type of a finite product of Eilenberg Mac Lane spaces, and then appealing to Zabrodsky's Theorem 2.1.

Let me conclude by mentioning one aspect of Miller's theorem which fails to generalize. Since Theorem 1 can be regarded as saying that every map from $\Sigma^k BG$ to Y is null-homotopic, for every k , it seemed natural to wonder if the same conclusion would hold if $\Sigma^k BG$ were replaced by the Thom space of a vector bundle over BG . After all, a Thom space is something like a twisted suspension. One does not have to look very far to see that the answer is no. Let P_n denote infinite real projective space with the subspace $\mathbb{R}P^{n-1}$ collapsed to a point. Then P_n is the Thom space of an n -plane bundle over $\mathbb{R}P^\infty$. Moreover, it is easy to check that $\mathrm{Hom}_{\mathcal{A}}(\bar{H}_*(\Sigma^t P_n), M) = 0$ for every bounded $\mathcal{A}$ -module M , just as in Lemma 1.15 of [6]. Nevertheless there is an obvious map $P_n \rightarrow \Sigma \mathbb{R}P^{n-1}$ which is essential.

REFERENCES

1. A. K. Bousfield and D. M. Kan, *Homotopy Limits, Completions and Localizations*, Lecture Notes in Math. **304**, Springer, 1972. MR **51**:1825
2. F. R. Cohen, T. J. Lada, and J. P. May, *The homology of iterated loop spaces*, Lecture Notes in Math. **533**, Springer, 1976. MR **55**:9096
3. E. Friedlander and G. Mislin, *Locally finite approximations of Lie groups*, *Inventiones Math.* **83** (1986) 425–436. MR **87i**:55038
4. B. Gray and C. A. McGibbon, *Universal phantom maps*, *Topology* **32** (1993) 371–394. MR **94a**:55008
5. C. A. McGibbon, *Phantom maps*, a chapter in *The Handbook of Algebraic Topology*, I. M. James, ed., North Holland, 1995.
6. H. Miller, *The Sullivan conjecture on maps from classifying spaces*, *Annals of Math.* **120** (1984) 39–87. MR **85i**:55012
7. S. B. Priddy, *On $\Omega^\infty S^\infty$ and the infinite symmetric group*, *Proc. Symp. Pure Math.* AMS **22** (1971) 217–220. MR **50**:11226
8. D. Sullivan, *Genetics of homotopy theory and the Adams conjecture*, *Annals of Math.* **100** (1974) 1–79. MR **56**:1305
9. A. Zabrodsky, *On phantom maps and a theorem of H. Miller*, *Israel J. Math.* **58** (1987) 129–143. MR **88m**:55028

DEPARTMENT OF MATHEMATICS, WAYNE STATE UNIVERSITY, DETROIT, MICHIGAN 48202
E-mail address: mcgibbon@math.wayne.edu