AGAIN NICE EQUATIONS FOR NICE GROUPS

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(Communicated by Ronald M. Solomon)

ABSTRACT. Nice quartinomial equations are given for unramified coverings of the affine line in nonzero characteristic p with $\mathrm{PSU}(2m-1, q')$ and $\mathrm{SU}(2m-1, q')$ as Galois groups where $m > 1$ is any integer and $q' > 1$ is any power of p .

1. INTRODUCTION

Let $m > 1$ be any integer, let $q' > 1$ be any power of a prime p , let $q = q'^2$, consider the polynomials $F^\dagger = F^\dagger(Y) = Y^{n'} + X^{q'}Y^v + XY^w + 1$ and $F'^* = F'^*(Y) = Y^{n'^*} + XY + 1$ in indeterminates X, Y over an algebraically closed field k of characteristic p , where $n' = 1 + q + \cdots + q^{2m-2}$, $v = 1 + q + \cdots + q^{m-1}$, $w = 1 + q + \cdots + q^{m-2}$, $n'^* = 1 + q' + \cdots + q'^{m-1}$, and consider their respective Galois groups $\mathrm{Gal}(F^\dagger, k(X))$ and $\mathrm{Gal}(F'^*, k(X))$. Both these are special cases of the families of polynomials giving unramified coverings of the affine line in nonzero characteristic which were written down in my 1957 paper [A01]. In my “Nice Equations” paper [A04], as a consequence of Cameron-Kantor Theorem I [CKa] on antiflag transitive collineation groups, I proved that $\mathrm{Gal}(F'^*, k(X))$ = the projective special linear group $\mathrm{PSL}(\nu, q')$; the $m = 2$ case of this was actually proved in my Feit-Serre-Email paper [A03] as a consequence of the Zassenhaus-Feit-Suzuki Theorem. In the present paper, as a consequence of Liebeck’s characterization of classical groups by orbit sizes [Li2], I shall show that $\mathrm{Gal}(F^\dagger, k(X))$ = the projective special unitary group $\mathrm{PSU}(2m-1, q')$. Note that Liebeck’s orbit size characterization depends on the Rank 3 characterization of Liebeck [Li1] and the primitive divisor characterization of Penttila-Praeger-Saxl [PPS], which in turn are based on CT = the Classification Theorem of Finite Simple Groups. Also note that, in the present paper, I only use the two-orbit case of Liebeck’s orbit size characterization which, as Liebeck points out in the Introduction of [Li2], depends only on Liebeck’s 1987 paper [Li1] and not on the Penttila-Praeger-Saxl paper [PPS].

As a corollary of the above-mentioned theorem that the Galois group of $F^\dagger$ is $\mathrm{PSU}(2m-1, q')$, I shall show that the Galois group of a more general polynomial $f^\dagger$ is also $\mathrm{PSU}(2m-1, q')$. Moreover, by slightly changing $f^\dagger$ and $F^\dagger$, I shall show that we get polynomials $\phi^\dagger$ and $\phi_1^\dagger$ whose Galois group is the special unitary group $\mathrm{SU}(2m-1, q')$. The polynomials $f^\dagger, \phi^\dagger$ and $\phi_1^\dagger$ are also special cases of the families of

Received by the editors March 21, 1995.

1991 *Mathematics Subject Classification*. Primary 12F10, 14H30, 20D06, 20E22.

This work was partly supported by NSF grant DMS 91-01424 and NSA grant MDA 904-92-H-3035.

polynomials giving unramified coverings of the affine line in nonzero characteristic written down in [A01].

It is a pleasure to thank Martin Liebeck for having promptly produced [Li2] at my request, and Ulrich Meierfrankenfeld for inspiring conversations.

2. NOTATION AND OUTLINE

Let k_p be a field of characteristic $p > 0$, let $q' > 1$ be any power of p , let $q = q'^2$, and let $m > 1$ be any integer. To abbreviate frequently occurring expressions, for every integer $i \geq -1$ we put

$$\langle i \rangle = 1 + q + q^2 + \cdots + q^i \quad (\text{convention: } \langle 0 \rangle = 1 \text{ and } \langle -1 \rangle = 0).$$

We shall frequently use the geometric series identity

$$1 + Z + Z^2 + \cdots + Z^i = \frac{Z^{i+1} - 1}{Z - 1}$$

and its corollary

$$\langle i \rangle = 1 + q + q^2 + \cdots + q^i = \frac{q^{i+1} - 1}{q - 1}.$$

Let

$$f^\dagger = f^\dagger(Y) = Y^{\langle 2m-2 \rangle} + 1 + \sum_{i=1}^{m-1} \left(T_i^{q'q^{i-1}} Y^{\langle m-2+i \rangle} + T_i Y^{\langle m-1-i \rangle} \right)$$

and note that then $f^\dagger$ is a monic polynomial of degree $\langle 2m-2 \rangle = 1 + q + q^2 + \cdots + q^{2m-2}$ in Y with coefficients in the polynomial ring $k_p[T_1, \dots, T_{m-1}]$. Now the constant term of $f^\dagger$ is 1 and the Y -exponent of every other term in $f^\dagger$ is 1 modulo p , and hence $f^\dagger - Y f_Y^\dagger = 1$ where $f_Y^\dagger$ is the Y -derivative of $f^\dagger$. Therefore $\text{Disc}_Y(f^\dagger) = 1$ where $\text{Disc}_Y(f^\dagger)$ is the Y -discriminant of $f^\dagger$, and hence the Galois group $\text{Gal}(f^\dagger, k_p(T_1, \dots, T_{m-1}))$ is well-defined as a subgroup of the symmetric group $\text{Sym}_{\langle 2m-2 \rangle}$.

For $1 \leq e \leq m-1$, let $f_e^\dagger$ be obtained by substituting $T_i = 0$ for all $i > e$ in $f^\dagger$, i.e., let

$$f_e^\dagger = f_e^\dagger(Y) = Y^{\langle 2m-2 \rangle} + 1 + \sum_{i=1}^e \left(T_i^{q'q^{i-1}} Y^{\langle m-2+i \rangle} + T_i Y^{\langle m-1-i \rangle} \right)$$

and note that then $f_e^\dagger$ is a monic polynomial of degree $\langle 2m-2 \rangle = 1 + q + q^2 + \cdots + q^{2m-2}$ in Y with coefficients in the polynomial ring $k_p[T_1, \dots, T_e]$ and, as above, $\text{Disc}_Y(f_e^\dagger) = 1$ and the Galois group $\text{Gal}(f_e^\dagger, k_p(T_1, \dots, T_e))$ is a subgroup of $\text{Sym}_{\langle 2m-2 \rangle}$. Note that if $k = k_p$ is an algebraically closed field (of characteristic $p > 0$), then $F^\dagger$ is obtained by substituting X for T_1 in $f_1^\dagger$ and hence $\text{Gal}(F^\dagger, k(X)) = \text{Gal}(f_1^\dagger, k_p(T_1))$.

In Section 3, we factor $f^\dagger$ as $f^\dagger = \bar{f} f^*$ where $\bar{f} = \bar{f}(Y)$ and $f^* = f^*(Y)$ are monic polynomials of degrees $(q'q^{m-1} + 1)\langle m-2 \rangle$ and $q^{m-1}(q\langle m-2 \rangle - q'\langle m-2 \rangle + 1)$ in Y with coefficients in $k_p[T_1, \dots, T_{m-1}]$ respectively. In Section 4, we show that $\bar{f}$ and f^* are irreducible in $k_p(T_1, \dots, T_{m-1})[Y]$, and hence $\text{Gal}(f^\dagger, k_p(T_1, \dots, T_{m-1}))$ may be regarded as a subgroup of $\text{PGL}(2m-1, q)$ having 2 orbits of sizes $(q'q^{m-1} + 1)\langle m-2 \rangle$ and $q^{m-1}(q\langle m-2 \rangle - q'\langle m-2 \rangle + 1)$. Given any e with $1 \leq e \leq m-1$, by putting $T_i = 0$ for all $i > e$ in $\bar{f}$ and f^* we get $f_e^\dagger = \bar{f}_e f_e^*$ where $\bar{f}_e$ and f_e^* are monic polynomials of degrees $(q'q^{m-1} + 1)\langle m-2 \rangle$

and $q^{m-1}(q\langle m-2\rangle - q'\langle m-2\rangle + 1)$ in Y with coefficients in $k_p[T_1, \dots, T_e]$ respectively. In Section 4, we also show that $\bar{f}_e$ and f_e^* are irreducible in $k_p(T_1, \dots, T_e)[Y]$, and hence $\text{Gal}(f_e^\dagger, k_p(T_1, \dots, T_e))$ may be regarded as a subgroup of $\text{PGL}(2m-1, q)$ having 2 orbits of sizes $(q'q^{m-1} + 1)\langle m-2\rangle$ and $q^{m-1}(q\langle m-2\rangle - q'\langle m-2\rangle + 1)$. In Section 6, from this orbit description, we deduce the result that if k_p is algebraically closed, then $\text{Gal}(f_e^\dagger, k_p(T_1, \dots, T_{m-1})) = \text{Gal}(f_e^\dagger, k_p(T_1, \dots, T_e)) = \text{PSU}(2m-1, q')$ for $1 \leq e \leq m-1$.

Consider the monic polynomials

$$\phi^\dagger = \phi^\dagger(Y) = Y^{q^{2m-1}-1} + 1 + \sum_{i=1}^{m-1} \left(T_i^{q'q^{i-1}} Y^{q^{m-1+i}-1} - T_i Y^{q^{m-i}-1} \right)$$

and

$$\begin{aligned} \phi_e^\dagger = \phi_e^\dagger(Y) &= Y^{q^{2m-1}-1} + 1 \\ &+ \sum_{i=1}^e \left(T_i^{q'q^{i-1}} Y^{q^{m-1+i}-1} - T_i Y^{q^{m-i}-1} \right) \quad \text{for } 1 \leq e \leq m-1 \end{aligned}$$

of degree $q^{2m-1} - 1$ in Y with coefficients in $k_p[T_1, \dots, T_{m-1}]$ and $k_p[T_1, \dots, T_e]$ respectively, and note that, as before, $\text{Disc}_Y(\phi^\dagger) = \text{Disc}_Y(\phi_e^\dagger) = 1$. In Section 6, as a consequence of the above result about the Galois groups of $f^\dagger$ and $f_e^\dagger$, we show that if k_p is algebraically closed, then $\text{Gal}(\phi^\dagger, k_p(T_1, \dots, T_{m-1})) = \text{Gal}(\phi_e^\dagger, k_p(T_1, \dots, T_e)) = \text{SU}(2m-1, q')$ for $1 \leq e \leq m-1$.

In Section 5, we give a review of linear algebra including definitions of $\text{PSU}(2m-1, q')$ and $\text{SU}(2m-1, q')$.

3. FACTORIZATION OF THE BASIC EQUATION

We find a root $h_m(Y) \in \text{GF}(p)[Y]$ of the polynomial

$$Y^{1+(q-q')\langle m-2\rangle} R^{q'} + R - \left(Y^{\langle 2m-2\rangle} + 1 \right)$$

by telescopically putting

$$h_m(Y) = \sum_{\mu=0}^{m-1} Y^{\alpha(m,\mu)} - \sum_{\mu=0}^{m-2} Y^{\alpha'(m,\mu)},$$

where

$$\alpha(m, \mu) = (q'q^{m-1} + 1)\langle m-2-\mu\rangle \quad \text{for } 0 \leq \mu \leq m-1$$

and

$$\alpha'(m, \mu) = (q^m + 1)\langle m-3-\mu\rangle + q^{m-2-\mu}[1 + (q-q')\langle \mu\rangle] \quad \text{for } 0 \leq \mu \leq m-2,$$

and checking that then

$$\begin{aligned} &1 + (q-q')\langle m-2\rangle + q'\alpha(m, 0) \\ &= 1 + (q-q')\langle m-2\rangle + q'(q'q^{m-1} + 1)\langle m-2\rangle \\ &= \langle 2m-2\rangle \end{aligned}$$

and, for $0 \leq \mu < m-1$,

$$\begin{aligned}
 & 1 + (q - q')\langle m-2 \rangle + q'\alpha(m, \mu+1) \\
 &= 1 + (q - q')\langle m-2 \rangle + q'(q'q^{m-1} + 1)\langle m-3-\mu \rangle \\
 &= 1 + q\langle m-2 \rangle + q^m\langle m-3-\mu \rangle - q'\langle m-2 \rangle + q'\langle m-3-\mu \rangle \\
 &= q^{m-2-\mu}(1 + q\langle \mu \rangle) + (q^m + 1)\langle m-3-\mu \rangle - q'q^{m-2-\mu}\langle \mu \rangle \\
 &= (q^m + 1)\langle m-3-\mu \rangle + q^{m-2-\mu}[1 + (q - q')\langle \mu \rangle] \\
 &= \alpha'(m, \mu)
 \end{aligned}$$

and, for $0 \leq \mu < m-1$,

$$\begin{aligned}
 & 1 + (q - q')\langle m-2 \rangle + q'\alpha'(m, \mu) \\
 &= 1 + (q - q')\langle m-2 \rangle + q'(q^m + 1)\langle m-3-\mu \rangle + q'q^{m-2-\mu}[1 + (q - q')\langle \mu \rangle] \\
 &= \langle m-2-\mu \rangle + q'[-\langle m-2 \rangle + (q^m + 1)\langle m-3-\mu \rangle + q^{m-2-\mu}\langle \mu+1 \rangle] \\
 &= (q'q^{m-1} + 1)\langle m-2-\mu \rangle \\
 &= \alpha(m, \mu)
 \end{aligned}$$

and

$$\alpha(m, m-1) = 0$$

and hence

$$\begin{aligned}
 & Y^{1+(q-q')\langle m-2 \rangle} h_m(Y)^{q'} + h_m(Y) \\
 &= \sum_{\mu=0}^{m-1} Y^{1+(q-q')\langle m-2 \rangle + q'\alpha(m, \mu)} - \sum_{\mu=0}^{m-2} Y^{1+(q-q')\langle m-2 \rangle + q'\alpha'(m, \mu)} \\
 &\quad + \sum_{\mu=0}^{m-1} Y^{\alpha(m, \mu)} - \sum_{\mu=0}^{m-2} Y^{\alpha'(m, \mu)} \\
 &= Y^{\langle 2m-2 \rangle} + 1.
 \end{aligned}$$

Likewise, for any integer $0 < i < m$, we find a root $h_i(Y, T_i) \in \text{GF}(p)[Y, T_i]$ of the polynomial

$$Y^{1+(q-q')\langle m-2 \rangle} R^{q'} + R - \left(T_i^{q'q^{i-1}} Y^{\langle m-2+i \rangle} + T_i Y^{\langle m-1-i \rangle} \right)$$

by telescopically putting

$$h_i(Y, T_i) = \sum_{\mu=0}^{i-1} Y^{\alpha(i, \mu)} T_i^{q^{i-1-\mu}} - \sum_{\mu=0}^{i-2} Y^{\alpha'(i, \mu)} T_i^{q'q^{i-2-\mu}},$$

where

$$\alpha(i, \mu) = q^{i-1-\mu}\langle m-1-i \rangle + (q'q^{m-1} + 1)\langle i-2-\mu \rangle \quad \text{for } 0 \leq \mu \leq i-1$$

and

$$\begin{aligned}
 \alpha'(i, \mu) &= \langle m-3-\mu \rangle \\
 &\quad + q^m\langle i-3-\mu \rangle + q^{m-2-\mu}[1 + (q - q')\langle \mu \rangle] \quad \text{for } 0 \leq \mu \leq i-2,
 \end{aligned}$$

and checking that then

$$\begin{aligned} & 1 + (q - q')\langle m - 2 \rangle + q'\alpha(i, 0) \\ &= 1 + (q - q')\langle m - 2 \rangle + q'q^{i-1}\langle m - 1 - i \rangle + q'(q'q^{m-1} + 1)\langle i - 2 \rangle \\ &= \langle m - 2 + i \rangle \end{aligned}$$

and, for $0 \leq \mu < i - 1$,

$$\begin{aligned} & 1 + (q - q')\langle m - 2 \rangle + q'\alpha(i, \mu + 1) \\ &= 1 + (q - q')\langle m - 2 \rangle + q'q^{i-2-\mu}\langle m - 1 - i \rangle + q'(q'q^{m-1} + 1)\langle i - 3 - \mu \rangle \\ &= 1 + q\langle m - 2 \rangle + q^m\langle i - 3 - \mu \rangle - q'\langle m - 2 \rangle + q'q^{i-2-\mu}\langle m - 1 - i \rangle + q'\langle i - 3 - \mu \rangle \\ &= \langle m - 3 - \mu \rangle + q^{m-2-\mu}(1 + q\langle \mu \rangle) + q^m\langle i - 3 - \mu \rangle - q'q^{m-2-\mu}\langle \mu \rangle \\ &= \langle m - 3 - \mu \rangle + q^m\langle i - 3 - \mu \rangle + q^{m-2-\mu}[1 + (q - q')\langle \mu \rangle] \\ &= \alpha'(i, \mu) \end{aligned}$$

and, for $0 \leq \mu < i - 1$,

$$\begin{aligned} & 1 + (q - q')\langle m - 2 \rangle + q'\alpha'(i, \mu) \\ &= 1 + (q - q')\langle m - 2 \rangle + q'\langle m - 3 - \mu \rangle \\ &\quad + q'q^m\langle i - 3 - \mu \rangle + q'q^{m-2-\mu}[1 + (q - q')\langle \mu \rangle] \\ &= \langle m - 2 - \mu \rangle + q'[-\langle m - 2 \rangle + \langle m - 3 - \mu \rangle + q^m\langle i - 3 - \mu \rangle + q^{m-2-\mu}\langle \mu + 1 \rangle] \\ &= q^{i-1-\mu}\langle m - 1 - i \rangle + (q'q^{m-1} + 1)\langle i - 2 - \mu \rangle \\ &= \alpha(i, \mu) \end{aligned}$$

and

$$\alpha(i, i - 1) = \langle m - 1 - i \rangle$$

and hence

$$\begin{aligned} & Y^{1+(q-q')\langle m-2 \rangle} h_i(Y, T_i)^{q'} + h_i(Y, T_i) \\ &= \sum_{\mu=0}^{i-1} Y^{1+(q-q')\langle m-2 \rangle + q'\alpha(i, \mu)} T_i^{q'q^{i-1-\mu}} - \sum_{\mu=0}^{i-2} Y^{1+(q-q')\langle m-2 \rangle + q'\alpha'(i, \mu)} T_i^{q'q^{i-1-\mu}} \\ &\quad + \sum_{\mu=0}^{i-1} Y^{\alpha(i, \mu)} T_i^{q^{i-1-\mu}} - \sum_{\mu=0}^{i-2} Y^{\alpha'(i, \mu)} T_i^{q'q^{i-2-\mu}} \\ &= T_i^{q'q^{i-1}} Y^{\langle m-2+i \rangle} + T_i Y^{\langle m-1-i \rangle}. \end{aligned}$$

Summing the above equations for h_i with $0 < i \leq m$ we get

$$Y^{1+(q-q')\langle m-2 \rangle} \bar{f}(Y)^{q'} + \bar{f}(Y) = f^\dagger(Y),$$

where we have put

$$\begin{aligned} \bar{f} = \bar{f}(Y) &= \sum_{\mu=0}^{m-1} Y^{\alpha(m, \mu)} - \sum_{\mu=0}^{m-2} Y^{\alpha'(m, \mu)} \\ &\quad + \sum_{i=1}^{m-1} \sum_{\mu=0}^{i-1} Y^{\alpha(i, \mu)} T_i^{q^{i-1-\mu}} - \sum_{i=1}^{m-1} \sum_{\mu=0}^{i-2} Y^{\alpha'(i, \mu)} T_i^{q'q^{i-2-\mu}}. \end{aligned}$$

By factoring the LHS of the previous equation, it follows that

$$f^\dagger = \bar{f}f^*, \quad \text{where} \quad f^* = f^*(Y) = Y^{1+(q-q')\langle m-2 \rangle} \bar{f}(Y)^{q'-1} + 1.$$

Note that the $(\mu = 0)$ term in the above first summation is $Y^{(q'q^{m-1}+1)\langle m-2 \rangle}$ and its exponent $(q'q^{m-1}+1)\langle m-2 \rangle$ is strictly greater than the Y -exponent of every other term in the above four summations, and hence $\bar{f}$ is a monic polynomial of degree $(q'q^{m-1}+1)\langle m-2 \rangle$ in Y with coefficients in $k_p[T_1, \dots, T_{m-1}]$, and therefore f^* is a monic polynomial of degree $1 + (q-q')\langle m-2 \rangle + (q'-1)(q'q^{m-1}+1)\langle m-2 \rangle = q^{m-1}[1 + (q-q')\langle m-2 \rangle]$ in Y with coefficients in $k_p[T_1, \dots, T_{m-1}]$. Thus

$$(3.0) \quad \begin{cases} f^\dagger = \bar{f}f^*, \text{ where } \bar{f} \text{ and } f^* \text{ are monic polynomials} \\ \text{of degrees } (q'q^{m-1}+1)\langle m-2 \rangle \text{ and } q^{m-1}[1 + (q-q')\langle m-2 \rangle] \text{ in } Y \\ \text{with coefficients in } k_p[T_1, \dots, T_{m-1}] \text{ respectively.} \end{cases}$$

For $1 \leq e \leq m-1$, let $\bar{f}_e = \bar{f}_e(Y)$ and $f_e^* = f_e^*(Y)$ be obtained by putting $T_i = 0$ for all $i > e$ in $\bar{f}$ and f^* respectively. Then by (3.0),

$$(3.1) \quad \begin{cases} \text{for } 1 \leq e \leq m-1 \text{ we have } f_e^\dagger = \bar{f}_e f_e^*, \text{ where } \bar{f}_e \text{ and } f_e^* \text{ are} \\ \text{monic polynomials of degrees} \\ (q'q^{m-1}+1)\langle m-2 \rangle \text{ and } q^{m-1}(q\langle m-2 \rangle - q'\langle m-2 \rangle + 1) \text{ in } Y \\ \text{with coefficients in } k_p[T_1, \dots, T_e] \text{ respectively.} \end{cases}$$

4. IRREDUCIBILITY

Now for $1 \leq e \leq m-1$ we have

$$f_e^\dagger = A_e T_1^{q'} - B_e T_1 + C_e,$$

where $0 \neq A_e = Y^{\langle m-1 \rangle} \in k_p[Y]$, $0 \neq B_e = -Y^{\langle m-2 \rangle} \in k_p[Y]$ and

$$C_e = Y^{\langle 2m-2 \rangle} + 1 + \sum_{i=2}^e \left(T_i^{q'q^{i-1}} Y^{\langle m-2+i \rangle} + T_i Y^{\langle m-1-i \rangle} \right) \in k_p[Y, T_2, \dots, T_e],$$

and hence in particular $\deg_{T_1} f_e^\dagger = q'$. Also clearly $\deg_{T_1} \bar{f}_e = 1$ and hence $\deg_{T_1} f_e^* = q' - 1$.

By letting I to be the Y -adic valuation of $Q = k_p(Y, T_2, \dots, T_e)$, i.e., the real discrete valuation whose valuation ring is the localization of $k_p[Y, T_2, \dots, T_e]$ at the principal prime ideal generated by Y , we see that $I(A_e) = \langle m-1 \rangle$ and $I(B_e) = \langle m-2 \rangle$; hence $I(B_e/A_e) = \langle m-2 \rangle - \langle m-1 \rangle = -q^{m-1}$ and therefore $\text{GCD}(q'-1, I(B_e/A_e)) = 1$. Also obviously A_e and C_e have no nonconstant common factor in $k_p[Y, T_2, \dots, T_e]$. Therefore by Lemmas (4.2) and (4.3) of [A05],

$$(4.1) \quad \bar{f}_e \text{ and } f_e^* \text{ are irreducible in } k_p(T_1, \dots, T_e)[Y] \text{ for } 1 \leq e \leq m-1.$$

By taking $e = m-1$ in (4.1) we see that,

$$(4.2) \quad \bar{f} \text{ and } f^* \text{ are irreducible in } k_p(T_1, \dots, T_{m-1})[Y].$$

Notation. Recall that $<$ denotes subgroup, and $\triangleleft$ denotes normal subgroup. Let the groups $\text{SL}(m, q) \triangleleft \text{GL}(m, q)$ and $\text{PSL}(m, q) \triangleleft \text{PGL}(m, q)$ and their actions on $\text{GF}(q)^m$ and $\mathcal{P}(\text{GF}(q)^m)$ be as on pages 78–80 of [A03]. Let

$$\Theta_m : \text{GL}(m, q) \rightarrow \text{PGL}(m, q) = \text{GL}(m, q)/\text{GF}(q)^*$$

be the canonical epimorphism where we identify the multiplicative group $\text{GF}(q)^*$ with scalar matrices which constitute the center of $\text{GL}(m, q)$.

Now, in view of Proposition 3.1 of [A04], by (3.0), (3.1), (4.1) and (4.2) we get the following:

Theorem (4.3). *Assuming $\text{GF}(q) \subset k_p$, for $1 \leq e \leq m-1$, in a natural manner we may regard*

$$\text{Gal}(\phi_e^\dagger, k_p(T_1, \dots, T_e)) < \text{GL}(2m-1, q)$$

and

$$\text{Gal}(f_e^\dagger, k_p(T_1, \dots, T_e)) < \text{PGL}(2m-1, q),$$

and then

$$\Theta_{2m-1}(\text{Gal}(\phi_e^\dagger, k_p(T_1, \dots, T_e))) = \text{Gal}(f_e^\dagger, k_p(T_1, \dots, T_e))$$

and $\text{Gal}(f_e^\dagger, k_p(T_1, \dots, T_e))$ has two orbits of sizes $(q'q^{m-1} + 1)\langle m-2 \rangle$ and $q^{m-1}(q\langle m-2 \rangle - q'\langle m-2 \rangle + 1)$. In particular, again assuming $\text{GF}(q) \subset k_p$, in a natural manner we may regard

$$\text{Gal}(\phi^\dagger, k_p(T_1, \dots, T_{m-1})) < \text{GL}(2m-1, q)$$

and

$$\text{Gal}(f^\dagger, k_p(T_1, \dots, T_{m-1})) < \text{PGL}(2m-1, q)$$

and then

$$\Theta_{2m-1}(\text{Gal}(\phi^\dagger, k_p(T_1, \dots, T_{m-1}))) = \text{Gal}(f^\dagger, k_p(T_1, \dots, T_{m-1}))$$

and $\text{Gal}(f^\dagger, k_p(T_1, \dots, T_{m-1}))$ has two orbits of sizes $(q'q^{m-1} + 1)\langle m-2 \rangle$ and $q^{m-1}(q\langle m-2 \rangle - q'\langle m-2 \rangle + 1)$.

Recall that a *quasi- p group* is a finite group which is generated by its p -Sylow subgroups. Since $\text{Disc}_Y f_e^\dagger = 1 = \text{Disc}_Y \phi_e^\dagger$ for $1 \leq e \leq m-1$, by the techniques of the proofs of Proposition 6 of [A01] and Lemma 34 of [A02] we get the following:

Theorem (4.4). *If k_p is algebraically closed, then, for $1 \leq e \leq m-1$, $\text{Gal}(f_e^\dagger, k_p(T_1, \dots, T_e))$ and $\text{Gal}(\phi_e^\dagger, k_p(T_1, \dots, T_e))$ are quasi- p groups. Hence in particular, if k_p is algebraically closed, then $\text{Gal}(f^\dagger, k_p(T_1, \dots, T_{m-1}))$ and $\text{Gal}(\phi^\dagger, k_p(T_1, \dots, T_{m-1}))$ are quasi- p groups.*

5. REVIEW OF LINEAR ALGEBRA

Dickson (page 131 of [Dic]) defines the *hyperorthogonal group* in $\text{GF}(q)$ on m indices as the group of all $a = (a_{ij}) \in \text{GL}(m, q)$ which leave the m -variate form

$$x_1^{q'+1} + \dots + x_m^{q'+1}$$

unchanged, i.e., for which

$$\sum_{j=1}^m \left(\sum_{i=1}^m x_i a_{ij} \right)^{q'+1} = \sum_{i=1}^m x_i^{q'+1}$$

or equivalently (page 133 of [Dic])¹

$$\sum_{j=1}^m a_{ij}^{q'+1} = 1 \quad \text{for } 1 \leq i \leq m$$

and

$$\sum_{j=1}^m a_{ij} a_{lj}^{q'} = 0 \quad \text{for } 1 \leq i \leq m \text{ and } 1 \leq l \leq m \text{ with } i \neq l.$$

Dickson denotes this group by $G_{m,p,s}$, where $p^s = q'$, and calculates (page 134 of [Dic]) its order $\Omega_{m,p,s}$; Dickson allows $m = 1$ and notes that (page 137 of [Dic]) then it is a cyclic group of order $q' + 1$. In modern terminology, this group is called the *general unitary group* and is denoted by $\text{GU}(m, q')$; see [LiK] where on the second line of Table 2.1C on page 19, Dickson's $\Omega_{m,p,s}$ is given as the order $|I|$. We also put $\text{SU}(m, q') = \text{GU}(m, q') \cap \text{SL}(m, q)$ and we call this the *special unitary group*; Dickson denotes this (page 137 of [Dic]) by $H_{m,p,s}$. Finally, we put $\text{PGU}(m, q') = \Theta_m(\text{GU}(m, q'))$ and $\text{PSU}(m, q') = \Theta_m(\text{SU}(m, q'))$, and we call these the *projective general unitary group* and *projective special unitary group* respectively; Dickson (page 138 of [Dic]) denotes $\text{PSU}(m, q')$ by $HO(m, q)$ and notes its simplicity provided $(m, q') \neq (2, 2), (2, 3), (3, 2)$ (note that we are always assuming $m > 1$).

Note that for any $H < \text{GL}(m, q)$ we have

$$(5.1) \quad \text{SU}(m, q') < H \Leftrightarrow \text{PSU}(m, q') < \Theta_m(H).$$

In case $(m, q') \neq (3, 2)$, this follows exactly as in the proof of Lemma 2.3 of [A04] because then by (22.4) of [Asc] $\text{SU}(m, q')$ is generated by transvections. Now the order of every transvection is p or 1, and the said proof is based on the fact that the group is generated by elements of p -power order, i.e., equivalently, the fact that it is a quasi- p group. So (5.1) holds also for $(m, q') = (3, 2)$; namely, $\text{SU}(3, 2)$ is a quasi-2 group because its transvections generate a subgroup of index 4 (see lines 13–14 on page 124 of [Tay]).

By (2.3.3), 2.10.4(ii) and 2.10.6(i) of [LiK], for any $H < \text{GL}(m, q)$ we have

$$(5.2) \quad \text{SU}(m, q') \triangleleft H \Leftrightarrow \text{SU}(m, q') < H < \text{GU}(m, q')\text{GF}(q)^*$$

and by 2.1.C of [LiK] we have

$$(5.3) \quad [\text{GU}(m, q')\text{GF}(q)^* : \text{SU}(m, q')] \not\equiv 0 \pmod{p}.$$

Since $\text{SU}(m, q)$ is quasi- p , it is generated by the p -power elements of $\text{SU}(m, q')\text{GF}(q)^*$, and hence these two subgroups have the same normalizer in $\text{GL}(m, q)$. Therefore by (5.2), for any $G < \text{PGL}(m, q)$ we have

$$(5.4) \quad \text{PSU}(m, q') \triangleleft G \Leftrightarrow \text{PSU}(m, q') < G < \text{PGU}(m, q')$$

and by (5.3) we get

$$(5.5) \quad [\text{PGU}(m, q') : \text{PSU}(m, q')] \not\equiv 0 \pmod{p}.$$

Finally, for any $H < \text{GL}(m, q)$ we obviously have

$$(5.6) \quad H < \text{GU}(m, q')\text{GF}(q)^* \Leftrightarrow \Theta_m(H) < \text{PGU}(m, q').$$

¹To make up for Dickson's unusual definition of the product of matrices (pages 76 and 88 of [Dic]), in his matrix (α_{ij}) , the index i should be regarded as the column number and j the row number.

In view of (5.4), Theorem (a) of [Li2] may be stated thus:

Theorem (5.7) [Liebeck]. $G < PGL(2m-1, q)$ has two orbits of sizes $(q'q^{m-1}+1)\langle m-2 \rangle$ and $q^{m-1}(q\langle m-2 \rangle - q'\langle m-2 \rangle + 1)$ if and only if after a suitable change of basis of $GF(q)^{2m-1}$ we have $PSU(2m-1, q') < G < PGU(2m-1, q')$.

Let $PSU(2m-1, q')_1$ denote $PSU(2m-1, q')$ as it acts on the orbit of size $(q'q^{m-1}+1)\langle m-2 \rangle$, and let $PSU(2m-1, q')_2$ denote $PSU(2m-1, q')$ as it acts on the orbit of size $(q'q^{m-1}+1)\langle m-2 \rangle$. These actions are faithful for $(m, q) \neq (2, 4)$ because $PSU(2m-1, q')$ is simple, and for $(m, q) = (2, 4)$ because the proper normal subgroups of $PSU(3, 2)$ have index 2, 4 or 8 (page 124 of [Tay]), and hence

$$(5.8) \quad PSU(2m-1, q')_1 \approx PSU(2m-1, q') \approx PSU(2m-1, q')_2$$

where $\approx$ denotes isomorphism of abstract groups.

6. GALOIS GROUPS

By (4.3), (5.1), (5.6) and (5.7) we get the following:

Theorem (6.1). If $GF(q) \subset k_p$, then, for $1 \leq e \leq m-1$, in a natural manner we have

$$SU(2m-1, q') < Gal(\phi_e^\dagger, k_p(T_1, \dots, T_e)) < GU(2m-1, q')GF(q)^*$$

and

$$PSU(2m-1, q') < Gal(f_e^\dagger, k_p(T_1, \dots, T_e)) < PGU(2m-1, q').$$

Hence in particular, if $GF(q) \subset k_p$, then in a natural manner we have

$$SU(2m-1, q') < Gal(\phi^\dagger, k_p(T_1, \dots, T_e)) < GU(2m-1, q')GF(q)^*$$

and

$$PSU(2m-1, q') < Gal(f^\dagger, k_p(T_1, \dots, T_e)) < PGU(2m-1, q').$$

By (3.0) to (3.1), (4.1), (4.2), (4.4), (5.2), (5.3), (5.4), (5.5), (5.8) and (6.1) we get the following:

Theorem (6.2). If k_p is algebraically closed, then, for $1 \leq e \leq m-1$, in a natural manner we have

$$Gal(\phi^\dagger, k_p(T_1, \dots, T_{m-1})) = Gal(\phi_e^\dagger, k_p(T_1, \dots, T_e)) = SU(2m-1, q')$$

and

$$Gal(f^\dagger, k_p(T_1, \dots, T_{m-1})) = Gal(f_e^\dagger, k_p(T_1, \dots, T_e)) = PSU(2m-1, q')$$

and

$$\begin{aligned} Gal(\bar{f}, k_p(T_1, \dots, T_{m-1})) &= Gal(\bar{f}_e, k_p(T_1, \dots, T_e)) \\ &= PSU(2m-1, q')_1 \approx PSU(2m-1, q') \end{aligned}$$

and

$$\begin{aligned} Gal(f^*, k_p(T_1, \dots, T_{m-1})) &= Gal(f_e^*, k_p(T_1, \dots, T_e)) \\ &= PSU(2m-1, q')_2 \approx PSU(2m-1, q'). \end{aligned}$$

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