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GOWERS' DICHOTOMY FOR ASYMPTOTIC STRUCTURE

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Abstract. In this paper Gowers' dichotomy is extended to the context of weaker forms of unconditionality, most notably asymptotic unconditionality. A general dichotomic principle is demonstrated; a Banach space has either a subspace with some unconditionality property, or a subspace with a corresponding 'proximity of subspaces' property.

0. Notation

In this paper, unless stated otherwise, all spaces will be infinite dimensional. Once we have a basis $\{x_i\}_{i=1}^{\infty}$ in a space X, we define a *finite support vector* to be a vector of the form $\sum_{i=n}^{m} a_i x_i$. The range of a finite support vector x will be the smallest interval $[n,m]$ such that x may be written as $\sum_{i=n}^{m} a_i x_i$. We write $x < y$ for two finite support vectors x and y if the range of x ends before the range of y begins (i.e. $\mathrm{supp}(x) = [n_1, m_1]$, $\mathrm{supp}(y) = [n_2, m_2]$ and $m_1 < n_2$). We say that the vectors $\{y_i\}_{i=1}^{k}$ are consecutive if $y_1 < y_2 < \cdots < y_k$.

A block subspace of X (with respect to a basis) is a subspace generated by a basic sequence of consecutive finite support vectors.

Finally, define an H.I. (Hereditarily Indecomposable) space to be a Banach space, in which two infinite dimensional subspaces have zero angle between them (i.e. for every $\varepsilon > 0$ and for all $Y, Z \subset X$, subspaces, there are vectors $y \in Y$, $z \in Z$, $\|y\| = \|z\| = 1$, $\|y - z\| < \varepsilon$). This also means that the span of any two disjoint infinite dimensional closed subspaces is not a closed subspace of X. The existence of such a space was recently proved in [GM].

1. Introduction

This paper is an application of the ideas of [G] to the language of asymptotic structure introduced in [MMiT].

The theorem at hand is the following:

Theorem 1.1 (Gowers)**.** *Every Banach space X contains a subspace with an unconditional basis, or an H.I. subspace.*

This theorem is based on a combinatorial result which concerns the following game. In a space X with a basis define $\Sigma(X) = \{\{x_i\}_{i=1}^{n};\ x_i$ are consecutive

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finite support vectors, $n \in \mathbb{N}$, $\|x_i\|_X = 1\}$. Choose $B \subseteq \Sigma(X)$. The game begins as player S chooses a block subspace X_1 in X. Then player V chooses $x_1 \in X_1$ normalized with finite support. Player S chooses another block subspace of X, X_2. Player V chooses $x_2 \in X_2$ with norm 1 and finite support such that $x_1 < x_2$. Then player S chooses another block subspace in which V chooses another vector with the same properties and so on and so forth. If, for some n, $\{x_i\}_{i=1}^n \in B$, then player V wins. If this never happens, player S wins. We say that there is a winning strategy for B in X if player V has a winning strategy for this game; that is, if $\forall X_1 \,\exists x_1 \,\forall X_2 \,\exists x_2 > x_1 \,\forall X_3 \,\exists x_3 > x_2 \cdots \exists n$ such that $\{x_i\}_{i=1}^n \in B$, where all X_i are block subspaces of X and all x_i are finite support normalized vectors.

Regarding this game, the following theorem holds.

Theorem 1.2 (Gowers)**.** *Given a space X with a basis, $B \subseteq \Sigma(X)$ and $\Delta = (\delta_1, \delta_2, \dots)$, $\delta_i > 0$, there exists $Y \subseteq X$, a block subspace, for which either $\Sigma(Y) \subseteq B^c$ or there is a winning strategy in Y for $(B)_\Delta$ (where $B^c = \Sigma(X) \backslash B$, $(B)_\Delta = \{\{x_i\}_{i=1}^n \in \Sigma(X);\, n \in \mathbb{N},\, \exists \{y_i\}_{i=1}^n \in B,\, \|x_i - y_i\| \leq \delta_i\}$).*

The above language is used to define the notion of asymptotic structure of an infinite dimensional Banach space. This notion was introduced in [MMiT].

Definition 1.3. Let X be a space with a basis $\{e_i\}_{i=1}^\infty$. A finite dimensional space F with a normalized basis $\{f_i\}_{i=1}^n$ is *asymptotic* in X, if for every $\varepsilon > 0$ player V has a winning strategy in X for $B_\varepsilon = \{\{x_i\}_{i=1}^n \subseteq \Sigma_n(X);\, \{x_i\}_{i=1}^n \overset{1+\varepsilon}{\sim} \{f_i\}_{i=1}^n\}$, as long as player S is confined to choosing tail subspaces (spaces of the form: $\mathrm{Span}\{e_i\}_{i=n}^\infty$).

Particularly, X is asymptotically unconditional if the normalized bases $\{f_i\}$ of all its asymptotic spaces are C-unconditional for some C. This notion was introduced and used earlier, in [MiSh].

It can be shown that there always exists a subspace, all further subspaces of which share the same asymptotic structure. Moreover, it follows from remarks in [C] and [MMiT], combined with Theorem 1.2, that in X, one may always find a block subspace Y, where playing the asymptotic game (S chooses tail subspaces), playing Gowers' game (S chooses any subspaces) or playing the game where S must choose the same tail subspace in every step will all yield the exact same set of asymptotic spaces for Y.

We say then that Y has *stabilized asymptotic structure*.

In this paper the proof method of Theorem 1.1 is applied to as. (asymptotic) unconditionality instead of unconditionality. Since as. unconditionality is much weaker than unconditionality, we prove a stronger alternative to it. Instead of just obtaining that every two subspaces have arbitrarily close vectors on their unit spheres, we show that we may choose vectors which are both "uniformly simple" in some sense, and arbitrarily close. Thus we get a stronger "proximity" property between subspaces, as an alternative to as. unconditionality. We continue by considering some weaker variations of unconditionality and get results in the same spirit.

Of course, one should ask if spaces which do not contain as. unconditional subspaces exist at all. The space constructed in [GM], and in stronger ways the spaces in [H] and [OS], are not asymptotically unconditional. In fact, the space constructed in [OS] does not have any of the unconditionality properties which will be discussed in this paper.

I would like to express my thanks to Professor E. Odell for pointing out to me the differences and relations between different variants of unconditionality (see section 3), as well as for other helpful comments; and to my advisor, Professor V. D. Milman, for advice and orientation.

2. Main result

Theorem 2.1. *Every Banach space X with a basis has one of the following properties*:

1. *X contains a subspace with an asymptotically unconditional basis.*
2. *X contains a block subspace X' with the following property*: *for every $\varepsilon > 0$ there exists an n, such that every $Y, Z \subset X'$, block subspaces, contain*

$$y_1 < z_1 < y_2 < z_2 < \cdots < y_n < z_n, \quad \text{such that: } y_i \in Y,\ z_i \in Z,$$

and

$$(*) \qquad \left\| \frac{\sum_{i=1}^n y_i}{\|\sum_{i=1}^n y_i\|} - \frac{\sum_{i=1}^n z_i}{\|\sum_{i=1}^n z_i\|} \right\| < \varepsilon.$$

Remarks. 1. Theorem 1.1 may be written in the following manner: Every Banach space X with a basis has one of the following properties:

1. X contains an unconditional basic sequence.
2. X contains a block subspace X' where the following holds: for every $\varepsilon > 0$, every $Y, Z \subset X'$, block subspaces, contain $y_1 < z_1 < y_2 < z_2 < \cdots < y_n < z_n$, such that $y_i \in Y$, $z_i \in Z$, and $\|\frac{\sum y_i}{\|\sum y_i\|} - \frac{\sum z_i}{\|\sum z_i\|}\| < \varepsilon$.

The difference between this result and our Theorem 2.1 lies in the number of y_i's and z_i's. If the alternative is unconditionality, n depends on ε, Y and Z. If it is as. unconditionality, n depends only on ε. This means that subspaces of such a space are "closer", in some sense, than subspaces of H.I. as. unconditional Banach spaces.

2. Note that in every Banach space with a basis, for any blocks: $y_1 < z_1 < \cdots < y_n < z_n$, one has: $\|\frac{\sum^n y_i}{\|\sum^n y_i\|} - \frac{\sum^n z_i}{\|\sum^n z_i\|}\| \geq \frac{1}{2Kn}$ (where K is the basic constant). Therefore, the restriction on the number of vectors cannot be improved to a bound independent of ε.

3. If X is an H.I. space with a basis, X' may be taken equal to X.

4. In fact there is a winning strategy in X' for vectors $y_1 < z_1 < \cdots < y_n < z_n$, with the property $\|\frac{\sum^n y_i}{\|\sum^n y_i\|} - \frac{\sum^n z_i}{\|\sum^n z_i\|}\| < \varepsilon$. Therefore, by passing to a subspace with stabilized asymptotic structure, the sequences $\{y_i, z_i\}_{i=1}^n$ may always be chosen to span spaces arbitrarily close to *asymptotic spaces* of X.

Proof of Theorem 2.1. Renorm X to have a bimonotone basis. Properties 1 and 2 in the theorem are stable under renorming, therefore it suffices to prove the theorem for the renormed space.

If for some block subspace Z, there exists K such that the normalized bases, $[f_i]_{i=1}^{2n}$, of all asymptotic spaces have

$$(1) \qquad \left\| \sum_{i=1}^{2n} (-1)^i \alpha_i f_i \right\| \leq K \left\| \sum_{i=1}^{2n} \alpha_i f_i \right\| \quad \forall \{\alpha_i\}_{i=1}^{2n} \subseteq \mathbb{R},$$

then Z is asymptotically unconditional. Indeed, remember that every normalized block sequence of the basis $\{f_i\}$ of any asymptotic space spans in turn an asymptotic space. Therefore (1) is true for the basis $\{f_i\}_{i=1}^{2n}$ and for all its normalized blocks, making the space K-unconditional.

If such K does not exist, in a block subspace X' with stabilized as. structure, for every K there exists $n''(K)$, and an asymptotic space $[g_i]_{i=1}^{2n''(K)}$ satisfying

$$\left\| \sum_{i=1}^{2n''(K)} (-1)^i \beta_i g_i \right\| > K \left\| \sum_{i=1}^{2n''(K)} \beta_i g_i \right\| \quad \text{for some } \{\beta_i\}_{i=1}^{2n} \subseteq \mathbb{R}. \tag{2}$$

Since the asymptotic structure is stabilized, in every block subspace and for all K there are vectors 2-equivalent to the $[g_i]_{i=1}^{n''(2K)}$ from (2) with $2K$ and $n''(2K)$. In particular those vectors will satisfy (2) for K and $n'(K) = n''(2K)$.

Now, in every block subspace and for all K we have vectors satisfying (2) for $2K$ and $n'(2K)$. By Theorem 1.2, in a further subspace there is a winning strategy for vectors which are δ-close to such vectors, for arbitrarily small δ. Choosing δ small enough, we can insure that the vectors produced by this strategy satisfy (2) with K and $n(K) = n'(2K)$.

When this strategy is played along with the subspace strategy $Y, Z, Y, Z, \ldots$ where $Y, Z \subset X'$ are block subspaces, the game yields a sequence of vectors $y_1 < z_1 < \cdots < y_{n(K)} < z_{n(K)}$ such that $y_i \in Y$, $z_i \in Z$ and for some $\{\beta_i\}_{i=1}^n, \{\beta_i'\}_{i=1}^n \subseteq \mathbb{R}$ one has:

$$\left\| \sum_{i=1}^{n(K)} \beta_i y_i + \sum_{i=1}^{n(K)} \beta_i' z_i \right\| \geq K \left\| \sum_{i=1}^{n(K)} \beta_i y_i - \sum_{i=1}^{n(K)} \beta_i' z_i \right\|.$$

We will show in Lemma 4.1 that this means:

$$\left\| \frac{\sum_{i=1}^{n(K)} \beta_i y_i}{\| \sum_{i=1}^{n(K)} \beta_i y_i \|} - \frac{\sum_{i=1}^{n(K)} \beta_i' z_i}{\| \sum_{i=1}^{n(K)} \beta_i' z_i \|} \right\| < \varepsilon_K \quad (\varepsilon_K \to 0 \text{ as } K \to \infty).$$

Note that if X is H.I., by [G], Remark 6, we may take X' equal to X. This proves Remark 3.

3. Variations on the main result

We outline in this section some slight variations of the result in Section 2, connected with different notions of unconditionality. These variations may appear insignificant at first sight; however, an example at the end of this section, noted to the author by E. Odell, suggests the remarkable conclusion that they are, indeed, meaningful.

3.1. In the proof of the main result, replace (1) by alternate quasi-unconditionality, by which we mean:

$$\left\| \sum_{i=1}^{2n} (-1)^i f_i \right\| \leq K \left\| \sum_{i=1}^{2n} f_i \right\|. \tag{1$'$}$$

Then (2) is replaced by

$$\left\| \sum_{i=1}^{2n} (-1)^i g_i \right\| > K \left\| \sum_{i=1}^{2n} g_i \right\|. \tag{2$'$}$$

Using the same argument one has the following result:

Theorem 3.1. *Every Banach space X contains a subspace with a basis X', with one of the following properties*:

1. *X' is an asymptotic alternately quasi-unconditional space (meaning that the normalized basis of any asymptotic space has property $(1)'$).*
2. *X' has property $(*)$ (as in Theorem 2.1), and additionally $\|y_i\| = \|z_i\| = 1$.*

3.2. If we replace $(1)'$ by

$$(1)'' \qquad \left\|\sum_{i=1}^{2n}(-1)^i \alpha_i f_i\right\| \leq K \left\|\sum_{i=1}^{2n} \alpha_i f_i\right\| \quad \text{for all } 1 \leq \alpha_i \leq 3,$$

then $(2)'$ becomes

$$(2)'' \qquad \left\|\sum_{i=1}^{2n}(-1)^i \beta_i g_i\right\| > K \left\|\sum_{i=1}^{2n} \beta_i g_i\right\| \quad \text{for some } 1 \leq \beta_i \leq 3.$$

Using Lemma 4.2 from Section 4 we see that the resulting dichotomy is:

Theorem 3.2. *Every Banach space X contains a subspace with a basis X', with one of the following properties:*

1. *X' is an asymptotic quasi-unconditional space (meaning that the normalized basis $[f_i]_{i=1}^n$, of any asymptotic space satisfies quasi-unconditionality: $\|\sum_{i=1}^n \varepsilon_i f_i\| \leq \|\sum_{i=1}^n f_i\|$, $\forall \varepsilon_i = \pm 1$).*
2. *X' has property $(*)$ (as in Theorem 2.1), and additionally $1 \leq \|y_i\|, \|z_i\| \leq 3$.*

3.3. The same technique may be applied to the original Gowers' dichotomy, Theorem 1.1 (formulated as in Remark 1). This results in the following variations on Gowers' Theorem:

Theorem 3.3. *Every Banach space X contains a subspace with a basis X', with one of the following properties:*

1. *All its normalized block sequences are (alternately) quasi-unconditional.*
2. *For every ε, and for all $Y, Z \subset X'$, block subspaces, there are vectors $y_1 < z_1 < \cdots < y_n < z_n$ such that $y_i \in Y$, $z_i \in Z$,*

$$\left\|\frac{\sum_{i=1}^n y_i}{\|\sum y_i\|} - \frac{\sum_{i=1}^n z_i}{\|\sum z_i\|}\right\| < \varepsilon \quad \textit{and} \quad 1 \leq \|y_i\|, \|z_i\| \leq 3 \qquad (1 = \|y_i\| = \|z_i\|).$$

The main point here is to realize that there is an essential dichotomy between variations of global and asymptotic unconditionality on the one hand, and types of proximity between subspaces on the other.

Odell [O] has observed that asymptotic-ℓ_1 spaces without unconditional subspaces, such as the space constructed in [AD], contain quasi-unconditional non-unconditional spaces. Therefore, by the dichotomic nature of Theorem 3.3 in the alternately unconditional case (both alternatives of the dichotomy cannot be attained simultaneously by one space) the equal distribution of the y_i's and z_i's may not be achieved in such spaces. This gives meaning to the results of this section.

It is not yet clear whether quasi-unconditionality and alternate quasi-unconditionality coincide, and whether their asymptotic counterparts are different from each other and from asymptotic unconditionality. Some results regarding these questions have been achieved by Odell [O].

4. Auxiliary lemmas

Lemma 4.1. *Let $C > 1$. If*

$$\left\|\sum_{i=1}^{n} \alpha_i y_i + \sum_{i=1}^{n} \beta_i z_i\right\| \geq C \left\|\sum_{i=1}^{n} \alpha_i y_i - \sum_{i=1}^{n} \beta_i z_i\right\|,$$

then

$$\left\|\frac{\sum \alpha_i y_i}{\|\sum \alpha_i y_i\|} - \frac{\sum \beta_i z_i}{\|\sum \beta_i z_i\|}\right\| \leq \frac{4}{C}.$$

Proof. Let $u = \sum_{i=1}^{n} \alpha_i y_i$, $v = \sum_{i=1}^{n} \beta_i z_i$. The proof is simply a triangle-inequality calculation. Without loss of generality, assume $\|u\| \leq \|v\|$. We have

$$\begin{aligned}
\left\|\frac{u}{\|u\|} - \frac{v}{\|v\|}\right\| &\leq \left\|\frac{u}{\|u\|} - \frac{u}{\|v\|}\right\| + \left\|\frac{u}{\|v\|} - \frac{v}{\|v\|}\right\| \\
&= \|u\| \left|\frac{1}{\|u\|} - \frac{1}{\|v\|}\right| + \frac{1}{\|v\|}\|u - v\| \\
&= \frac{\|v\| - \|u\|}{\|v\|} + \frac{\|u - v\|}{\|v\|} \leq \frac{2\|u - v\|}{\|v\|} \\
&\leq \frac{2}{C}\frac{\|u + v\|}{\|v\|} \leq \frac{2}{C}\left(\frac{\|u\| + \|v\|}{\|v\|}\right) \\
&\leq \frac{4\|v\|}{C\|v\|} = \frac{4}{C}.
\end{aligned}$$

Lemma 4.2. *Let X have a bimonotone basis. If for every n, every $\{x_i\}_{i=1}^{2n} \subseteq X$ such that $n < x_1 < \cdots < x_{2n}$ and $1 \leq \|x_i\| \leq 3$ have*

$$(**) \qquad \left\|\sum_{i=1}^{2n} (-1)^i x_i\right\| \leq C \left\|\sum_{i=1}^{2n} x_i\right\|,$$

then every $\{y_i\}_{i=1}^{2n}$ such that $n < y_1 < \cdots < y_{2n}$, $\|y_i\| = 1$ have:

$$\left\|\sum_{i=1}^{2n} \varepsilon_i y_i\right\| \leq (4C + 5) \left\|\sum_{i=1}^{2n} y_i\right\| \quad \forall \varepsilon_i = \pm 1.$$

Hence X is as. quasi-unconditional.

Proof. Given ε_i, y_i define $K = \{1 \leq i \leq 2n; \varepsilon_i = -1\}$. Partition K alternately into K_1 and K_2:

$$\sum_{i=1}^{2n} \varepsilon_i y_i = \left(\sum_{i=1}^{2n} y_i - 2\sum_{i \in K_1} y_i\right) + \left(\sum_{i=1}^{2n} y_i - 2\sum_{i \in K_2} y_i\right) - \sum_{i=1}^{2n} y_i.$$

The first and second terms are $\sum_{i=1}^{2n} \varepsilon_i^{(j)} y_i$ ($j = 1, 2$), where $\varepsilon_i^{(j)} = \pm 1$, and no two consecutive $\varepsilon_i^{(j)}$'s are equal to -1.

Let $J^{(j)} = \{1 \leq i \leq 2n, \varepsilon_i^{(j)} = 1\}$, $j = 1, 2$. Partition $J^{(j)}$ into alternating $J_1^{(j)}, J_2^{(j)}$:

$$\sum_{i=1}^{2n} \varepsilon_i^{(j)} y_i = \left(-\sum_{i=1}^{2n} y_i + 2\sum_{i \in J_1^{(j)}} y_i\right) + \left(-\sum_{i=1}^{2n} y_i + 2\sum_{i \in J_2^{(j)}} y_i\right) + \sum_{i=1}^{2n} y_i.$$

The first and second terms are $\sum_{i=1}^{2n} \varepsilon_i^{(j,k)} y_i$ $(1 \le j, k \le 2)$, where $\varepsilon_i^{(j,k)} = \pm 1$, and no two consecutive $\varepsilon_i^{(j,k)}$'s equal 1, no four consecutive $\varepsilon_i^{(j,k)}$'s equal -1.

Therefore $\sum_{i=1}^{2n} \varepsilon_i^{(j,k)} y_i = \pm \sum_{i=1}^{l^{(j,k)}} (-1)^i z_i^{(j,k)}$ where $z_i^{(j,k)}$ is the sum of at most three consecutive y_i's (hence $1 \le \|z_i^{(j,k)}\| \le 3$) and $l^{(j,k)}$ is the number of z_i's (hence $l^{(j,k)} < 2n$). So:

$$\begin{aligned}
\left\| \sum_{i=1}^{2n} \varepsilon_i y_i \right\| &\le \sum_{j,k=1}^{2} \left\| \sum_{i=1}^{2n} \varepsilon_i^{(j,k)} y_i \right\| + \left\| \sum_{i=1}^{2n} y_i \right\| \\
&= \sum_{j,k=1}^{2} \left\| \sum_{i=1}^{l^{(j,k)}} (-1)^i z_i^{(j,k)} \right\| + \left\| \sum_{i=1}^{2n} y_i \right\| \\
&\le \sum_{j,k=1}^{2} (C+1) \left\| \sum_{i=1}^{l^{(j,k)}} z_i^{(j,k)} \right\| + \left\| \sum_{i=1}^{2n} y_i \right\| \\
&= (4(C+1)+1) \left\| \sum_{i=1}^{2n} y_i \right\| = (4C+5) \left\| \sum_{i=1}^{2n} y_i \right\| .
\end{aligned}$$

(Last inequality by $(**)$, modified for the possibility of an odd number of terms.)

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