ON MAXIMAL FUNCTIONS IN ORLICZ SPACES

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ABSTRACT. Let $\Phi(t)$ and $\Psi(t)$ be the functions having the representations $\Phi(t) = \int_0^t a(s)ds$ and $\Psi(t) = \int_0^t b(s)ds$, where $a(s)$ is a positive continuous function such that $\int_1^\infty \frac{a(s)}{s} ds = +\infty$ and $b(s)$ is quasi-increasing. Then the maximal function Mf is a function in Orlicz space L^Φ for all $f \in L^\Psi$ if and only if there exists a positive constant c_1 such that $\int_1^s \frac{a(t)}{t} dt \leq c_1 b(c_1 s)$ for all $s \geq 1$.

1. INTRODUCTION

Let T be the group of real numbers modulo 2π , and let $f(x)$ be a real valued integrable function defined on T with period 2π . The classical Hardy-Littlewood maximal function $Mf(x)$ is defined by

$$(1.1) \quad Mf(x) := \sup_{x \in I} \frac{1}{|I|} \int_I |f(y)| dy,$$

where the supremum is taken over all open intervals $I \subset T$ with $x \in I$.

The aim of this paper is to give a necessary and sufficient condition such that the Hardy-Littlewood maximal function $Mf(x)$ defined by (1.1) is a function in Orlicz space L^Φ , whenever $f(x)$ is an arbitrary function in Orlicz space L^Ψ . Orlicz space L^Ψ is defined as follows.

Definition 1.1. Let $\Psi(t)$ be a nondecreasing continuous function such that $\lim_{t \rightarrow \infty} \Psi(t) = +\infty$. Put

$$(1.2) \quad L^\Psi := \left\{ f : \int_0^{2\pi} \Psi(\varepsilon |f(x)|) dx < +\infty \text{ for some } \varepsilon > 0 \right\}.$$

Then the space L^Ψ is called an Orlicz space (see Kita and Yoneda [2], Rao and Ren [4] and Zygmund [6]).

We note that if $L^\Psi(t) = t^p$ for $t \geq 0$ and $p > 1$, then L^Ψ is a usual Lebesgue space.

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2. MAIN THEOREMS

As is well known, the sublinear operator M is of type (p, p) ($1 < p \leq \infty$) and weak type $(1, 1)$ (see Torchinsky [5] and Zygmund [6]). For the maximal functions in the class $\Phi(L) := \{f : \int_{\mathbf{R}^n} \Phi(|f(x)|)dx < +\infty\}$, the detailed results can be found in the monograph of Kokilashvili and Krbeć [3]. In this paper the maximal function of a function in Orlicz function space L^Ψ defined by (1.2) will be considered.

A function $b(s)$ defined on $[0, \infty)$ is called quasi-increasing if there exists a positive constant c_0 such that $b(s_1) \leq c_0 b(c_0 s_2)$ for all $0 < s_1 < s_2$.

Let $a(s)$ and $b(s)$ be positive continuous functions defined on $[0, \infty)$ satisfying the following properties:

$$(2.1) \quad \int_1^\infty \frac{a(s)}{s} ds = +\infty;$$

$$(2.2) \quad b(s) \text{ is quasi-increasing} \quad \text{and} \quad \lim_{s \rightarrow \infty} b(s) = +\infty.$$

Put

$$(2.3) \quad \Phi(t) := \int_0^t a(s)ds \quad \text{and} \quad \Psi := \int_0^t b(s)ds \quad \text{for} \quad t \geq 0.$$

We have the following result which is also a generalization of the result in [5], p.103.

Theorem 2.1. *Let $a(s)$, $b(s)$, $\Phi(t)$ and $\Psi(t)$ be the functions satisfying the above properties (2.1) – (2.3). Then the following statements are equivalent:*

(i) *there exists a positive constant c_1 such that*

$$(2.4) \quad \int_1^s \frac{a(t)}{t} dt \leq c_1 b(c_1 s) \quad \text{for all} \quad s \geq 1;$$

(ii) *there exists a positive constant c_2 such that*

$$(2.5) \quad \int_0^{2\pi} \Phi(Mf(x))dx \leq c_2 + c_2 \int_0^{2\pi} \Psi(c_2|f(x)|)dx \quad \text{for all} \quad f \in L^1(T).$$

As a corollary of Theorem 2.1 we have the following result.

Theorem 2.2. *Let $a(s)$, $b(s)$, $\Phi(t)$ and $\Psi(t)$ be the functions satisfying (2.1)–(2.3). Then the maximal function Mf is a function in Orlicz space L^Φ for all $f \in L^\Psi$ if and only if the functions $a(s)$ and $b(s)$ satisfy the inequality (2.4).*

We note that in Theorems 2.1 and 2.2 the functions $\Phi(t)$ and $\Psi(t)$ do not necessarily satisfy the Δ_2 -condition, that is, there exist positive constants c and t_0 such that $\Phi(2t) \leq c\Phi(t)$ for all $t \geq t_0$.

Let us point out two particular consequences of the above theorems.

Corollary 2.3. *Let $a(s)$ be a positive continuous function defined on $[0, \infty)$ such that $\int_1^\infty \frac{a(s)}{s} ds = +\infty$. Put $b(s) := \int_1^s \frac{a(t)}{t} dt + 1$ for $s \geq 1$, $b(s) = 1$ for $0 \leq s < 1$, $\Phi(t) := \int_0^t a(s)ds$ and $\Psi(t) := \int_0^t b(s)ds$. Then for any Orlicz space $L^{\Psi_1} \not\subseteq L^\Psi$, there exists a function $f_0 \in L^{\Psi_1}$ such that $f_0 \notin L^\Psi$ and $Mf_0 \notin L^\Phi$.*

Corollary 2.4. *Let $a(s)$ be a positive quasi-increasing continuous function defined on $[0, \infty)$. Put $\Phi(t) := \int_0^t a(s)ds$. Then the following statements are equivalent:*

(i) *there exists a positive constant c_1 such that*

$$(2.6) \quad \int_1^s \frac{a(t)}{t} dt \leq c_1 a(c_1 s) \quad \text{for all } s \geq 1.$$

(ii) *there exists a positive constant c_2 such that*

$$(2.7) \quad \int_0^{2\pi} \Phi(Mf(x)) dx \leq c_2 + c_2 \int_0^{2\pi} \Phi(c_2 |f(x)|) dx \quad \text{for all } f \in L^1.$$

3. PROOF OF THE THEOREMS

First we prove Theorem 2.1. Let us prove (i) $\Rightarrow$ (ii). Let $\chi_{\{Mf>1\}}(x)$ be a characteristic function on the set $\{x \in T : Mf(x) > 1\}$ and put $F(x) := \Phi(Mf(x))\chi_{\{Mf>1\}}(x)$. Then it follows that

$$\begin{aligned} I &:= \int_{Mf>1} \Phi(Mf(x)) dx = \int_0^{2\pi} F(x) dx = \int_0^\infty |\{F > \lambda\}| d\lambda \\ &= \int_0^{\Phi(1)} |\{F > \lambda\}| d\lambda + \int_{\Phi(1)}^\infty |\{F > \lambda\}| d\lambda \\ &\leq 2\pi\Phi(1) + \int_{\Phi(1)}^\infty |\{F > \lambda\}| d\lambda. \end{aligned}$$

From (2.1) and (2.3), $\Phi(t)$ is strictly increasing. Therefore we get

$$(3.1) \quad I \leq 2\pi\Phi(1) + \int_{\Phi(1)}^\infty |\{Mf > \Phi^{-1}(\lambda)\}| d\lambda.$$

Put $t = \Phi^{-1}(\lambda)$. From (2.1) and (2.3) $\lim_{t \rightarrow \infty} \Phi(t) = +\infty$ holds. Therefore we have

$$(3.2) \quad \int_{\Phi(1)}^\infty |\{F > \lambda\}| d\lambda = \int_1^\infty |\{Mf > t\}| a(t) dt.$$

Since the sublinear operator M is simultaneously of weak-type (1,1) and of type (∞, ∞) , it follows by the well known result (see Torchinsky [5], p.92) that there exist positive constants c_3 and c_4 such that

$$(3.3) \quad |\{Mf > t\}| \leq \frac{c_3}{t} \int_{t/c_4}^\infty |\{|f| > s\}| ds \quad \text{for all } t > 0.$$

Therefore it follows from (3.1), (3.2) and (3.3) that

$$\begin{aligned} I &\leq 2\pi\Phi(1) + c_3 \int_1^\infty \frac{a(t)}{t} \left(\int_{t/c_4}^\infty |\{|f| > s\}| ds \right) dt \\ &= 2\pi\Phi(1) + c_3 \int_{1/c_4}^\infty \left(|\{|f| > s\}| \int_1^{c_4 s} \frac{a(t)}{t} dt \right) ds. \end{aligned}$$

By the assumption (i), we have

$$\begin{aligned} I &\leq 2\pi\Phi(1) + c_1 c_3 \int_{1/c_4}^\infty |\{|f| > s\}| b(c_1 c_4 s) ds \\ &\leq 2\pi\Phi(1) + (c_3/c_4) \int_0^{2\pi} \Psi(c_1 c_4 |f(x)|) dx. \end{aligned}$$

We conclude that (2.5) holds.

Next we prove (ii) $\Rightarrow$ (i). Let (ii) hold and we assume that (i) does not hold. Then there exists a sequence of numbers $\{s_k : k \geq 1\}$ such that $s_k \geq 1$ for $k \geq 1$ and

$$(3.4) \quad \int_1^{s_k} \frac{a(t)}{t} dt > 2^k b(k 2^k s_k) \quad \text{for } k \geq 1 .$$

We choose a collection of disjoint open intervals $\{I_k : k \geq k_0\}$, $I_k \subseteq T$, such that

$$(3.5) \quad |I_k| := \frac{1}{2^k \Psi(2^k s_k)} \quad \text{for } k \geq k_0 \quad \text{and} \quad \sum_{k=k_0}^{\infty} |I_k| < 2\pi .$$

Put

$$(3.6) \quad f(x) := \frac{\varepsilon_0}{c_2} \sum_{k=k_0}^{\infty} 2^k s_k \chi_{I_k}(x) , \quad 0 < \varepsilon_0 < 1 ,$$

where χ_{I_k} is a characteristic function on I_k and a number ε_0 will be defined later. Then it follows from (3.5) and (3.6) that

$$\begin{aligned} \int_0^{2\pi} \Psi(c_2 |f(x)|) dx &= \sum_{k=k_0}^{\infty} \int_{I_k} \Psi(c_2 |f(x)|) dx \\ &= \sum_{k=k_0}^{\infty} \Psi(\varepsilon_0 2^k s_k) |I_k| \leq \sum_{k=k_0}^{\infty} \Psi(2^k s_k) \cdot \frac{1}{2^k \Psi(2^k s_k)} \\ &= \sum_{k=k_0}^{\infty} 2^{-k} < +\infty . \end{aligned}$$

Therefore we get $f \in L^\Psi$. Since $b(s)$ is quasi-increasing, we have

$$\Psi(t) \geq \int_{t/2}^t b(s) ds \geq \frac{1}{c_0} \int_{t/2}^t b\left(\frac{t}{2c_0}\right) ds = \frac{t}{2c_0} b\left(\frac{t}{2c_0}\right) .$$

Therefore $L^\Psi \subseteq L^1$ holds and we can choose ε_0 such that

$$(3.7) \quad |f|_T := \frac{1}{2\pi} \int_0^{2\pi} |f(x)| dx < 1 .$$

On the other hand, for each $0 < \varepsilon < 1$, $\int_0^{2\pi} \Phi(\varepsilon M f(x)) dx = +\infty$ holds. Indeed, put $g = \varepsilon f$. It is a well known fact (see Torchinsky [5], p.93) that there exists a positive constant c_5 such that

$$(3.8) \quad |\{Mg > \lambda\}| \geq \frac{c_5}{\lambda} \int_{|g|>\lambda} |g(x)| dx \quad \text{for all } \lambda > |g|_T .$$

Therefore it follows from (3.7) and (3.8) that

$$\begin{aligned}
 \int_0^{2\pi} \Phi(\varepsilon M f(x)) dx &= \int_0^{2\pi} \Phi(M g(x)) dx = \int_0^\infty |\{M g > \lambda\}| \Phi'(\lambda) d\lambda \\
 &\geq \int_{|g|_T}^\infty |\{M g > \lambda\}| \Phi'(\lambda) d\lambda \geq \int_1^\infty |\{M g > \lambda\}| \Phi'(\lambda) d\lambda \\
 &\geq c_5 \int_1^\infty \left(\int_{|g|>\lambda} |g(x)| dx \right) \frac{\Phi'(\lambda)}{\lambda} d\lambda \\
 &= c_5 \int_1^\infty \left(\int_0^{2\pi} |g(x)| \chi_{\{|g|>\lambda\}}(x) dx \right) \frac{\Phi'(\lambda)}{\lambda} d\lambda \\
 &= c_5 \int_0^{2\pi} |g(x)| \left(\int_1^\infty \chi_{\{|g|>\lambda\}}(x) \frac{\Phi'(\lambda)}{\lambda} d\lambda \right) dx \\
 &= c_5 \int_{|g|>1} |g(x)| \left(\int_1^{|g(x)|} \frac{a(\lambda)}{\lambda} d\lambda \right) dx .
 \end{aligned}$$

Since $g(x) = \varepsilon f(x)$ and $f(x)$ is constant on I_k , we get

$$g(x) = \frac{\varepsilon \varepsilon_0}{c_2} 2^k s_k \geq \frac{\varepsilon \varepsilon_0 2^k}{c_2} \uparrow +\infty \quad \text{as} \quad k \uparrow +\infty .$$

Thus, there exists a positive integer $k_1 > k_0$ such that

$$(3.9) \quad \frac{\varepsilon \varepsilon_0 2^k}{c_2} > 1 \quad \text{for all} \quad k \geq k_1 .$$

From (3.4), (3.5) and (3.9), it follows that

$$\begin{aligned}
 \int_0^{2\pi} \Phi(\varepsilon M f(x)) dx &\geq c_5 \sum_{k=k_1}^\infty \int_{I_k} |g(x)| \left(\int_1^{|g(x)|} \frac{a(\lambda)}{\lambda} d\lambda \right) dx \\
 &= \frac{c_5 \varepsilon \varepsilon_0}{c_2} \sum_{k=k_1}^\infty 2^k s_k \left(\int_1^{\frac{\varepsilon \varepsilon_0 2^k s_k}{c_2}} \frac{a(\lambda)}{\lambda} d\lambda \right) |I_k| \\
 &\geq \frac{c_5 \varepsilon \varepsilon_0}{c_2} \sum_{k=k_1}^\infty 2^k s_k \left(\int_1^{s_k} \frac{a(\lambda)}{\lambda} d\lambda \right) \cdot \frac{1}{2^k \Psi(2^k s_k)} \\
 &\geq \frac{c_5 \varepsilon \varepsilon_0}{c_2} \sum_{k=k_1}^\infty 2^k s_k \cdot 2^k s_k b(k 2^k s_k) \cdot \frac{1}{2^k \Psi(2^k s_k)} .
 \end{aligned}$$

Since $b(s)$ is quasi-increasing, it follows that

$$\Psi(2^k s_k) = \int_0^{2^k s_k} b(s) ds \leq \int_0^{2^k s_k} c_0 b(c_0 2^k s_k) ds = c_0 2^k s_k b(c_0 2^k s_k) .$$

Therefore we have

$$\begin{aligned}
 \int_0^{2\pi} \Phi(\varepsilon M f(x)) dx &\geq \frac{c_5 \varepsilon \varepsilon_0}{c_2} \sum_{k=k_1}^\infty \frac{2^k s_k \cdot 2^k b(k 2^k s_k)}{2^k \cdot c_0 2^k s_k \cdot b(c_0 2^k s_k)} \\
 &= \frac{c_5 \varepsilon \varepsilon_0}{c_2 c_0} \sum_{k=k_1}^\infty \frac{b(k 2^k s_k)}{b(c_0 2^k s_k)} .
 \end{aligned}$$

If we choose a positive integer $k_2 > k_1$ such that $k_2 \geq c_0^2$, then $b(c_0 2^k s_k) \leq c_0 b(c_0 \cdot \frac{k}{c_0} 2^k s_k)$ for all $k \geq k_2$. Therefore we have

$$\int_0^{2\pi} \Phi(\varepsilon M f(x)) dx \geq \frac{c_5 \varepsilon \varepsilon_0}{c_2 c_0^2} \sum_{k=k_2}^{\infty} 1 = +\infty.$$

We arrive at a contradiction and Theorem 2.1 is proved.

Last we prove Theorem 2.2. Let us assume that the inequality (2.4) holds. Let f be an arbitrary function in Orlicz space L^Ψ . Then there exists a positive number ε_1 such that $\int_0^{2\pi} \Psi(\varepsilon_1 |f(x)|) dx < +\infty$. By Theorem 2.1, we have the following inequality:

$$\int_0^{2\pi} \Phi\left(\frac{\varepsilon_1}{c_2} M f(x)\right) dx \leq c_2 + c_2 \int_0^{2\pi} \Psi(\varepsilon_1 |f(x)|) dx < +\infty.$$

Thus $Mf \in L^\Phi$ holds.

Conversely, let us assume that (2.4) does not hold. Then, by the same manner as the proof of Theorem 2.1, it follows that there exists a function $f \in L^\Psi$ such that $Mf \notin L^\Phi$. This is a contradiction and the proof is complete.

Remark. Let $a(s)$ be a positive quasi-increasing continuous function defined on $[0, \infty)$. Put $\Phi(t) = \int_1^t a(s) ds$ for $t \geq 1$ and $\Phi(t) = 0$ for $0 \leq t < 1$. It is easy to see from the result given in [3] (see p.6, Theorem 1.2.1) that the following statements are equivalent:

(i) there exists a positive constant c_1 such that

$$\int_1^t \frac{a(s)}{s} ds \leq c_1 a(c_1 t) \quad \text{for all } t \geq 1;$$

(ii) there exists a positive constant c_2 such that

$$\int_1^t \frac{\Phi(s)}{s^2} ds \leq \frac{c_2 \Phi(c_2 t)}{t} \quad \text{for all } t \geq 1;$$

(iii) there exists a constant $\ell > 1$ such that

$$\Phi(t) \leq \frac{1}{2\ell} \Phi(\ell t) \quad \text{for all } t \geq 1;$$

(iv) the function Φ^α is quasi convex on $[0, \infty)$ for some $0 < \alpha < 1$, that is, there exist a convex function ω and a constant $c > 0$ such that $\omega(t) \leq \Phi(t) \leq c\omega(ct)$ for all $t \geq 0$.

We note that if $a(t)$ is an increasing function on $[0, \infty)$ satisfying the inequality in the statement of (i), then there exist positive constants α , t_0 and c_0 such that $a(t) \geq c_0 t^\alpha$ for all $t \geq t_0$ (see Rao and Ren [4], p.26, Corollary 5).

4. EXAMPLES

In this section, some examples of functions $\Phi(t)$, $\Psi(t)$, $a(t)$ and $b(t)$ will be given. Let $\varphi_1(t)$ and $\varphi_2(t)$ be the functions defined on $[0, \infty)$. We write $\varphi_1(t) \sim \varphi_2(t)$, if there exist positive constants c_1, c_2 and t_0 such that $c_1 \varphi_1(t) \leq \varphi_2(t) \leq c_2 \varphi_1(t)$ for all $t \geq t_0$.

Example 1. Let $1 < p < +\infty$.

$$\begin{cases} \Phi(t) = \frac{1}{p}t^p, & a(t) = t^{p-1} \quad \text{for } t \geq 0; \\ \Psi(t) = \frac{1}{p}t^p, & b(t) = t^{p-1} \quad \text{for } t \geq 0. \end{cases}$$

Example 2. Let $0 < \alpha \leq 1$.

$$\begin{cases} \Phi(t) \sim \frac{t}{(\log t)^{1-\alpha}}, & a(t) \sim \frac{1}{(\log t)^{1-\alpha}}; \\ \Psi(t) \sim t(\log t)^\alpha, & b(t) \sim (\log t)^\alpha. \end{cases}$$

Example 3.

$$\begin{cases} \Phi(t) \sim \frac{t}{\log t}, & a(t) \sim \frac{1}{\log t} \\ \Psi(t) \sim t(\log \log t), & b(t) \sim \log \log t. \end{cases}$$

Example 4. Put $\log^+ t = \log t$ for $t \geq 1$ and $\log^+ t = 0$ for $0 \leq t < 1$. And put $L_1(t) = \log^+ t$, $L_n(t) = \log^+ L_{n-1}(t)$ for $n \geq 2$.

$$\begin{cases} \Phi(t) \sim \frac{t}{L_1(t)L_2(t) \cdots L_{n-1}(t)}, & a(t) \sim \frac{1}{L_1(t)L_2(t) \cdots L_{n-1}(t)} \\ \Psi(t) \sim tL_n(t), & b(t) \sim L_n(t). \end{cases}$$

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