ON THE FINITENESS OF BASS NUMBERS OF LOCAL COHOMOLOGY MODULES

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(Communicated by Wolmer V. Vasconcelos)

ABSTRACT. In this note we show that, if I is an ideal of dimension 1 of an analytically irreducible local ring, then the Bass numbers of local cohomology modules with support in $V(I)$ are finite.

1. INTRODUCTION

We assume that all rings are noetherian with identity throughout this note. Our aim in this note is to prove

Theorem. *Let R be an analytically irreducible local ring (see [19, chap. VIII]) with maximal ideal $\mathfrak{m}$, I an ideal of dimension 1 and M a finitely generated R -module. Then the Bass numbers of local cohomology modules $H_I^j(M)$ are finite for all $j \geq 0$.*

Let R be a ring, I an ideal of R , and N an R -module. We say N is I -cofinite if the support of N is contained in $V(I)$ and $\text{Ext}_R^i(R/I, N)$ is finitely generated for all $i \geq 0$ (cf. [5, EXPOSÉ XIII, 1.1] or [6, §2]).

If M is a finitely generated R -module, then the local cohomology modules $H_{\mathfrak{m}}^j(M)$ are well known to be Artinian ([13, Proposition 2.1]). The relationship between $\mathfrak{m}$ -cofiniteness and Artinian condition is given by the following remark which is a consequence of Matlis duality (cf. [14]) and the fundamental work of A. Grothendieck (cf. [4]):

Remark 1 (cf. [6]). Let R be a complete local ring with maximal ideal $\mathfrak{m}$ and residue field k and let N be an R -module. Then the following conditions are equivalent:

- (i) N satisfies the descending chain condition (dcc);
- (ii) N is isomorphic to a submodule of the finite direct sum of copies of the injective hull E of the residue field k of R ;
- (iii) there is a finitely generated R -module M such that $\text{Hom}_R(M, E) \simeq N$, and $\text{Hom}_R(N, E) \simeq M$;
- (iv) $\text{Supp}_R(N) = V(\mathfrak{m})$, and $\text{Hom}_R(k, N)$ is finitely generated;
- (v) $\text{Supp}_R(N) = V(\mathfrak{m})$, and $\text{Ext}_R^i(k, N)$ is finitely generated for all $i \geq 0$;
- (vi) M is $\mathfrak{m}$ -cofinite.

Received by the editors February 15, 1995 and, in revised form, April 13, 1995.

1991 *Mathematics Subject Classification.* Primary 14B15, 13D03, 18G15.

Key words and phrases. Local cohomology, cofinite, Bass number, analytically irreducible, Ext.

There are some questions (cf. [11, Conjecture 4.3, Conjecture 4.4 and Problem 4, etc]). Conjecture 4.4 has been solved affirmatively by C. Huneke and R. Y. Sharp (cf. [10, Theorem 2.1]) for regular local rings of characteristic p and by G. Lyubeznik (cf. [12, Corollary 3.6 (d)]) for those of characteristic zero containing a field. Hartshorne's counter-example [6, §3, An example] says that the above conjecture is not true for an ideal of dimension 2 without the regularity assumption.

2. PROOF OF THE LEMMAS

Lemma 1. *Let I be an ideal of R , and p a non-negative integer. Then, for any R -module T (not necessarily finitely generated), the following conditions are equivalent:*

- (i) $\text{Ext}_R^i(R/I, T)$ is finitely generated for all $i \leq p$;
- (ii) for any $P \in \text{Min}(R/I)$, $\text{Ext}_R^i(R/P, T)$ is finitely generated for all $i \leq p$;
- (iii) for any finitely generated R -module N with support in $V(I)$, $\text{Ext}_R^i(N, T)$ is finitely generated for all $i \leq p$,

where $\text{Min}(R/I)$ is the set of all minimal prime ideals of I .

Proof. The equivalence (i) $\Leftrightarrow$ (iii) has been already known (cf. [9, Lemma 4.2] or [3, Lemma 2]).

We show (i) $\Rightarrow$ (ii). Suppose that $\text{Ext}_R^i(R/I, T)$ is finitely generated for all $i \leq p$. Let I' be an arbitrary ideal of R containing I . Since the support of R/I' is contained in $V(I)$, $\text{Ext}_R^i(R/I', T)$ is finitely generated for all $i \leq p$ by the implication (i) $\Rightarrow$ (iii). In particular, for $P \in \text{Min}(R/I)$, $\text{Ext}_R^i(R/P, T)$ is finitely generated for all $i \leq p$.

Next we prove (ii) $\Rightarrow$ (i). We may assume that I is a radical ideal by the equivalence (i) $\Leftrightarrow$ (iii). Put $\text{Min}(R/I) = \{P_1, P_2, \dots, P_n\}$, and $I = P_1 \cap P_2 \cap \dots \cap P_n$. We prove (ii) $\Rightarrow$ (i) using induction on n . Suppose that $\text{Ext}_R^i(R/P_j, T)$ is finitely generated for all $i \leq p$ and all j such that $1 \leq j \leq n$, which is the case $n = 1$. Set $I_j = P_1 \cap P_2 \cap \dots \cap P_j$ and suppose that $\text{Ext}_R^i(R/I_j, T)$ is finitely generated for all $i \leq p$ and all j such that $1 \leq j < n$. The exact sequence

$$0 \longrightarrow I_{n-1}/I_{n-1} \cap P_n \longrightarrow R/I_{n-1} \cap P_n \longrightarrow R/I_{n-1} \longrightarrow 0$$

induces a long exact sequence

$$\begin{aligned} \cdots \longrightarrow \text{Ext}_R^i(R/I_{n-1}, T) &\longrightarrow \text{Ext}_R^i(R/I_{n-1} \cap P_n, T) \\ &\longrightarrow \text{Ext}_R^i(I_{n-1}/I_{n-1} \cap P_n, T) \longrightarrow \cdots \end{aligned}$$

Since $I_{n-1}/I_{n-1} \cap P_n \simeq (I_{n-1} + P_n)/P_n$, this is a finitely generated R -module with support in $V(P_n)$. Hence $\text{Ext}_R^i(R/I_{n-1}, T)$ and $\text{Ext}_R^i(I_{n-1}/I_{n-1} \cap P_n, T)$ are finitely generated for all $i \leq p$, by inductive hypothesis and the implication (i) $\Rightarrow$ (iii). Consequently $\text{Ext}_R^i(R/I, T)$ is finitely generated for all $i \leq p$. $\square$

Remark 2. We remark that if $(R, \mathfrak{m})$ is local and T is I -cofinite, then $\text{Ext}_R^i(R/\mathfrak{m}, T)$ is finitely generated for all $i \geq 0$ by Lemma 1. We use this fact in the sequel.

Remark 3. We remark the following facts since we need these to prove Lemma 2:

Let $R \rightarrow R'$ be a flat extension of rings, M and T R -modules, and I an ideal of R . Then

- (i) If M is finitely generated, then $\text{Ext}_R^i(M, T) \otimes_R R' \simeq \text{Ext}_{R'}^i(M \otimes_R R', T \otimes_R R')$ (see [15, (3.E)]).

- (ii) The local cohomology functor $H_I^j(-)$ commutes with the tensor functor $(-)\otimes_R R'$, that is $H_I^j(M)\otimes_R R' \simeq H_{IR'}^j(M\otimes_R R')$ for any R -module M (cf. [18, Proposition 4.1.5]).
- (iii) Suppose furthermore that $R \rightarrow R'$ is faithfully flat. Then $T\otimes_R R'$ is finitely generated as an R' -module if and only if T is finitely generated as an R -module (cf. [2, chap. I, §3, no. 6, Proposition 11]).
- (iv) If R is local, then $\dim R/I = \dim \widehat{R}/I\widehat{R}$, where $\widehat{R}$ is the completion with respect to the maximal ideal of R (cf. [16, §8]).

Lemma 2. *Let $R, \mathfrak{m}, I$ and M be as in the theorem. Let N be a finitely generated R -module whose support is contained in $V(\mathfrak{m})$. Then $\text{Ext}_R^i(N, H_I^j(M))$ is a finitely generated R -module for all $i \geq 0$ and all $j \geq 0$.*

Proof. Since R is an analytically irreducible local ring, we may assume that R is a local domain complete with the topology defined by $\mathfrak{m}$ by Remark 3. Further, we may assume that I is a radical ideal. We only prove the lemma for the case $N = R/\mathfrak{m}$ by Lemma 1. If R is equi-characteristic, then $H_I^j(M)$ is I -cofinite for all $j \geq 0$ by [3, Theorem 3]. Thus the lemma holds by Remark 2. Assume that R is not equi-characteristic. Let q be a uniformizing parameter of R . Put $\text{Min}(R/I) = \{P_1, P_2, \dots, P_n\}$, which is the set of minimal prime ideals of I , and we have $I = P_1 \cap P_2 \cap \dots \cap P_n$. We denote $\text{Min}(R/I)$ by S . Divide this set S into two parts: one set S_1 consisting of prime ideals of S containing q and the other S_2 not containing q . We set $S_1 = \{P_1, P_2, \dots, P_t\}$ and $S_2 = S \setminus S_1$, where t is an integer such that $0 \leq t \leq n$. Since the ideal I satisfies the condition of [3, Theorem 3] in the case $t = 0$ or n , $H_I^j(M)$ is I -cofinite for all $j \geq 0$. Thus the assertion holds by Remark 2. So we may assume $0 < t < n$. Put $I_1 = P_1 \cap \dots \cap P_t$ and $I_2 = P_{t+1} \cap \dots \cap P_n$, and we see that $V(I_1) \cup V(I_2) = V(I)$ and $V(I_1) \cap V(I_2) = V(\mathfrak{m})$. We have a Mayer-Vietoris exact sequence (cf. [8, Chap. III, Ex. 2.4]):

$$\dots \longrightarrow H_{\mathfrak{m}}^j(M) \longrightarrow H_{I_1}^j(M) \oplus H_{I_2}^j(M) \longrightarrow H_I^j(M) \longrightarrow \dots$$

We decompose this sequence into short exact sequences of kernels and cokernels as follows:

$$(\alpha)^j \quad 0 \longrightarrow X^j \longrightarrow H_{\mathfrak{m}}^j(M) \longrightarrow Y^j \longrightarrow 0,$$

$$(\beta)^j \quad 0 \longrightarrow Y^j \longrightarrow H_{I_1}^j(M) \oplus H_{I_2}^j(M) \longrightarrow Z^j \longrightarrow 0,$$

$$(\gamma)^j \quad 0 \longrightarrow Z^j \longrightarrow H_I^j(M) \longrightarrow X^{j+1} \longrightarrow 0.$$

We note that X^q and Y^q are Artinian for all $q \geq 0$, since $H_{\mathfrak{m}}^q(M)$ is Artinian for all $q \geq 0$.

Fix $j \geq 0$ arbitrarily. By the sequence $(\beta)^j$, we get an exact sequence:

$$\begin{aligned} \dots \longrightarrow \text{Ext}_R^i(R/\mathfrak{m}, H_{I_1}^j(M)) \oplus \text{Ext}_R^i(R/\mathfrak{m}, H_{I_2}^j(M)) \\ \longrightarrow \text{Ext}_R^i(R/\mathfrak{m}, Z^j) \longrightarrow \text{Ext}_R^{i+1}(R/\mathfrak{m}, Y^j) \longrightarrow \dots \end{aligned}$$

Again by virtue of [3, Theorem 3] and Remark 2, $\text{Ext}_R^i(R/\mathfrak{m}, H_{I_1}^j(M))$ and $\text{Ext}_R^i(R/\mathfrak{m}, H_{I_2}^j(M))$ are finitely generated for all $i \geq 0$. As $\text{Ext}_R^i(R/\mathfrak{m}, Y^j)$ is finitely generated, $\text{Ext}_R^i(R/\mathfrak{m}, Z^j)$ is finitely generated for all $i \geq 0$. Furthermore we get an exact sequence by $(\gamma)^j$:

$$\dots \longrightarrow \text{Ext}_R^i(R/\mathfrak{m}, Z^j) \longrightarrow \text{Ext}_R^i(R/\mathfrak{m}, H_I^j(M)) \longrightarrow \text{Ext}_R^i(R/\mathfrak{m}, X^{j+1}) \longrightarrow \dots$$

As X^{j+1} is also Artinian, $\text{Ext}_R^i(R/\mathfrak{m}, H_I^j(M))$ is finite for all $i \geq 0$ by the above sequence. We have completed the proof. $\square$

3. PROOF OF THE MAIN THEOREM AND COROLLARIES

Remark 4. We need the following facts for the proof of the main theorem and its corollaries.

- (i) Let $(R, \mathfrak{m})$ be local, $\widehat{R}$ its completion with respect to $\mathfrak{m}$, and T an R -module. If T has support only at $\mathfrak{m}$, then $T \otimes_R \widehat{R}$ has support only at $\mathfrak{m}\widehat{R}$.
- (ii) Let $R \rightarrow R'$ be a faithfully flat extension of rings, and T as above. If $T \otimes_R R'$ satisfies the descending chain condition as an R' -module, then T satisfies the descending chain condition as an R -module.

Proof of the main theorem. Let P be any prime ideal of R . We show that the Bass numbers (cf. [1]):

$$\mu^i(P, H_I^j(M)) = \dim_{\kappa(P)} \text{Ext}_{R_P}^i(\kappa(P), H_I^j(M)_P)$$

are finite for each P , where $\kappa(P) = R_P/PR_P$. If P does not contain I , then the local cohomology module $H_I^j(M)$ vanishes at P . Thus we may suppose that P contains I . Since I is an ideal of dimension 1, P is either a minimal prime ideal of I or the maximal ideal of R . If P is a minimal prime ideal of I , then I is PR_P -primary in R_P . Making R_P complete with respect to PR_P , $\text{Ext}_{R_P}^i(\kappa(P), H_I^j(M)_P) \otimes_{R_P} \widehat{R}_P$ is finitely generated as an $\widehat{R}_P$ -module by Remark 1, where $\widehat{R}_P$ is the completion of R_P defined by the PR_P -adic topology. Hence we have the theorem by Remark 3. If P is the maximal ideal, then the assertion follows from Lemma 2. $\square$

We have the following corollaries of the theorem (cf. [11]):

Corollary 1. *Let $R, \mathfrak{m}, I$ and M be as in the theorem. Then $\text{Soc}(H_I^j(M))$ is finitely generated for all $j \geq 0$.*

Proof. The assertion immediately follows from the theorem. $\square$

Corollary 2. *Let $R, \mathfrak{m}, I$ and M be as above. Let j be a non-negative integer. If the local cohomology module $H_I^j(M)$ has support only at $\mathfrak{m}$, then $H_I^j(M)$ is Artinian.*

Proof. By Remark 4, $H_I^j(M) \otimes_R \widehat{R}$ has support only at $\mathfrak{m}\widehat{R}$, where $\widehat{R}$ is the completion of R with respect to $\mathfrak{m}$. By Lemma 2, $H_I^j(M) \otimes_R \widehat{R}$ is $\mathfrak{m}\widehat{R}$ -cofinite. It follows from Remark 1 that $H_I^j(M) \otimes_R \widehat{R}$ is an Artinian $\widehat{R}$ -module. Since $\widehat{R}$ is a faithfully flat R -module, $H_I^j(M)$ satisfies the descending chain condition as an R -module by Remark 4, that is to say, $H_I^j(M)$ is Artinian. $\square$

ACKNOWLEDGMENT

The author is grateful to Professor Y. Hinohara for his valuable suggestions, advice and hearty support during work on this note.

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