THE BOUNDEDNESS OF RIESZ s -TRANSFORMS OF MEASURES IN $\mathbb{R}^n$

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ABSTRACT. Let μ be a finite nonzero Borel measure in $\mathbb{R}^n$ satisfying $0 < c^{-1}r^s \leq \mu B(x, r) \leq cr^s < \infty$ for all $x \in \text{spt } \mu$ and $0 < r \leq 1$ and some $c > 0$. If the Riesz s -transform

$$\mathcal{C}_{s,\mu}(x) = \int \frac{y-x}{|y-x|^{s+1}} d\mu y$$

is essentially bounded, then s is an integer. We also give a related result on the L^2 -boundedness.

1. INTRODUCTION

The Cauchy transform of a finite nonnegative Borel measure μ on $\mathbb{C}$ is the analytic function $\mathcal{C}_\mu$ defined in the complement of the support of μ , $\text{spt } \mu$, by

$$\mathcal{C}_\mu(z) = \int \frac{d\mu \zeta}{\zeta - z}.$$

Often problems in complex analysis can be reduced to the study of relations between $\mathcal{C}_\mu$ and the geometric properties of μ . In this paper we investigate the behavior of an s -variant $\mathcal{C}_{s,\mu}$ of the Riesz transform in $\mathbb{R}^n$,

$$\mathcal{C}_{s,\mu}(x) = \int \frac{y-x}{|y-x|^{s+1}} d\mu y,$$

where $0 < s < n$. This is defined for all $x \in \mathbb{R}^n$ for which

$$\int \frac{d\mu y}{|y-x|^s} < \infty.$$

This transform is called Riesz s -transform. In many cases $\mathcal{C}_{s,\mu}$ does not exist in the usual sense for $x \in \text{spt } \mu$ but it may exist as a principal value

$$\mathcal{C}_{s,\mu}(x) = \lim_{\epsilon \downarrow 0} \int_{\mathbb{R}^n \setminus B(x, \epsilon)} \frac{y-x}{|y-x|^{s+1}} d\mu y,$$

where $B(x, \epsilon)$ is the x -centered closed ball with radius ϵ .

David and Semmes (see [DS1, DS2]) have studied singular integrals on integral dimensional surfaces and rectifiable sets. $\mathcal{C}_{s,\mu}$ defines a bounded operator on $L^2(\mu) \rightarrow L^2(\mu)$ if s is an integer and μ is a surface measure on a sufficiently nice s -dimensional surface, e.g. a Lipschitz graph. In this case the principal values $\mathcal{C}_{s,\mu}(x)$ also exist μ a.e.

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Mattila and Preiss (see [MP]) have shown that if for μ almost all $x \in \mathbb{R}^n$ $0 < \liminf r^{-s} \mu B(x, r) \leq \limsup r^{-s} \mu B(x, r) < \infty$ and the principal values $\mathcal{C}_{s,\mu}(x)$ exist μ a.e., then s is an integer and μ is s -rectifiable, i.e. there exist s -dimensional C^1 submanifolds M_i such that $\mu(\mathbb{R}^n \setminus \cup_i M_i) = 0$. In this paper we show that the essential boundedness of $\mathcal{C}_{s,\mu}$ implies that s is an integer. As a consequence it follows that also the L^2 -boundedness forces s to be an integer. It is not known whether μ is s -rectifiable. David and Semmes have shown in [DS1] that the L^2 -boundedness of a large class of singular integral operators implies the rectifiability of μ , and it forces s to be an integer.

2. PRELIMINARY LEMMAS

Let μ be a finite Borel measure on $\mathbb{R}^n$, $\epsilon > 0$ and $x \in \mathbb{R}^n$. The truncated Riesz s -transform is defined by

$$\mathcal{C}_{s,\mu}^\epsilon(x) = \int_{\mathbb{R}^n \setminus B(x,\epsilon)} \frac{y-x}{|y-x|^{s+1}} d\mu y$$

and the maximal truncated transform by

$$\mathcal{C}_{s,\mu}^*(x) = \sup_{\epsilon > 0} |\mathcal{C}_{s,\mu}^\epsilon(x)|.$$

If μ is only locally finite, we shall use the modified maximal function

$$\tilde{\mathcal{C}}_{s,\mu}^*(x) = \sup_{0 < r < 1} \left| \int_{B(x,1) \setminus B(x,r)} \frac{y-x}{|y-x|^{s+1}} d\mu y \right|.$$

For a finite measure μ , $\tilde{\mathcal{C}}_{s,\mu}^*$ is bounded if and only if $\mathcal{C}_{s,\mu}^*$ is bounded. We also denote

$$\underline{D}^s(\mu, x) = \liminf_{r \downarrow 0} r^{-s} \mu B(x, r)$$

and

$$\overline{D}^s(\mu, x) = \limsup_{r \downarrow 0} r^{-s} \mu B(x, r).$$

The Riesz s -transform $\mathcal{C}_{s,\mu}$ has the following properties.

Lemma 1. *The Riesz s -transform $\mathcal{C}_{s,\mu}$ of a finite Borel measure μ is defined for $\mathcal{L}^n$ a.e. x and $\mathcal{C}_{s,\mu} \in L_{loc}^p$ for $1 \leq p < n/s$.*

The proof is straightforward and we omit it.

The proof of the following lemma was suggested by Stephen Semmes.

Lemma 2. *Suppose that μ is a finite Borel measure in $\mathbb{R}^n$ satisfying $\mu B(x, r) \leq r^s$ for all $x \in \mathbb{R}^n$ and $r > 0$. Then for all $x \in \mathbb{R}^n$*

$$\mathcal{C}_{s,\mu}^*(x) \leq \|\mathcal{C}_{s,\mu}\|_\infty + C,$$

where C is a constant depending on n and s and $\|\cdot\|_\infty$ is the L^∞ norm.

Proof. This is a variant of a well-known inequality of Cotlar (see e.g. [J]).

We give here the main steps of the proof. If $\|\mathcal{C}_{s,\mu}\|_\infty = \infty$ the result is clear. Suppose that $\|\mathcal{C}_{s,\mu}\|_\infty < \infty$.

Let $\epsilon > 0$ and $x \in \mathbb{R}^n$. Consider the mean value integral

$$\begin{aligned} \frac{1}{\alpha_n(\epsilon/2)^n} \int_{B(x, \epsilon/2)} \int_{B(x, \epsilon)} \frac{d\mu y}{|y-z|^s} d\mathcal{L}^n z &\leq \frac{1}{\alpha_n(\epsilon/2)^n} \int_{B(x, \epsilon)} \int_{B(0, 2\epsilon)} \frac{d\mathcal{L}^n z}{|z|^s} d\mu y \\ &\leq \frac{n}{n-s} 2^{2n-s}, \end{aligned}$$

where $\alpha_n = \mathcal{L}^n B(0, 1)$ is the Lebesgue measure of the unit ball in $\mathbb{R}^n$. Hence there is a point $z \in B(x, \epsilon/2)$ such that

$$(1) \quad \int_{B(x, \epsilon)} \frac{d\mu y}{|y-z|^s} \leq \frac{n}{n-s} 2^{2n-s}$$

and $|\mathcal{C}_{s, \mu}(z)| \leq \|\mathcal{C}_{s, \mu}\|_\infty$.

Now we can estimate

$$|\mathcal{C}_{s, \mu}^\epsilon(x) - \mathcal{C}_{s, \mu}(z)| \leq \int_{\mathbb{R}^n \setminus B(x, \epsilon)} \left| \frac{y-x}{|y-x|^{s+1}} - \frac{y-z}{|y-z|^{s+1}} \right| d\mu y + \int_{B(x, \epsilon)} \frac{d\mu y}{|y-z|^s}.$$

We estimate the first term on the right-hand side. Divide the set $\mathbb{R}^n \setminus B(x, \epsilon)$ into annuli $B(x, 2^{i+1}\epsilon) \setminus B(x, 2^i\epsilon)$, $i = 0, 1, 2, \dots$. Using the mean value theorem and the fact that $|D(x|x|^{-s-1})| \leq C|x|^{-s-1}$ we obtain

$$\begin{aligned} \int_{\mathbb{R}^n \setminus B(x, \epsilon)} \left| \frac{y-x}{|y-x|^{s+1}} - \frac{y-z}{|y-z|^{s+1}} \right| d\mu y &\leq \int_{\mathbb{R}^n \setminus B(x, \epsilon)} \left| D \frac{\xi(y)}{|\xi(y)|^{s+1}} \right| |x-z| d\mu y \\ (2) \quad &\leq C \sum_{i=0}^{\infty} \int_{B(x, 2^{i+1}\epsilon) \setminus B(x, 2^i\epsilon)} \frac{|x-z|}{|\xi(y)|^{s+1}} d\mu y \\ &\leq C 2^{2s+1}, \end{aligned}$$

where $\xi(y)$ is a point on the line segment between the points $y-x$ and $y-z$. Using the estimates (1) and (2) we get

$$|\mathcal{C}_{s, \mu}^\epsilon(x)| \leq C 2^{2s+1} + \frac{n}{n-s} 2^{2n-s} + \|\mathcal{C}_{s, \mu}\|_\infty$$

and so $\mathcal{C}_{s, \mu}^*(x) \leq \|\mathcal{C}_{s, \mu}\|_\infty + C 2^{2s+1} + \frac{n}{n-s} 2^{2n-s}$. $\square$

There is a useful variant of this lemma which states that under the assumptions on μ in Lemma 2 $\tilde{\mathcal{C}}_{s, \mu}^*$ is bounded everywhere if it is bounded on the support of μ .

Lemma 3. Suppose that μ is a locally finite Borel measure in $\mathbb{R}^n$ satisfying $\mu B(x, r) \leq r^s$ for all $x \in \mathbb{R}^n$ and $r > 0$. If $\tilde{\mathcal{C}}_{s, \mu}^*$ is bounded on the support of μ , then $\tilde{\mathcal{C}}_{s, \mu}^*$ is bounded on $\mathbb{R}^n$.

Proof. The proof is similar to the proof of the preceding lemma.

Let $x \in \mathbb{R}^n$ and $\epsilon > 0$. Denote $d = \text{dist}(x, \text{spt } \mu) = |x - z|$, $z \in \text{spt } \mu$. If $\epsilon \leq d < 1/2$ we have

$$\begin{aligned} \left| \int_{B(x,1) \setminus B(x,\epsilon)} \frac{y-x}{|y-x|^{s+1}} d\mu y \right| &\leq \left| \int_{B(z,2d)} \frac{y-x}{|y-x|^{s+1}} d\mu y \right| \\ &+ \left| \int_{(B(x,1) \setminus B(z,1)) \cup (B(z,1) \setminus B(x,1))} \frac{y-x}{|y-x|^{s+1}} d\mu y \right| \\ &+ \int_{B(z,1) \setminus B(z,2d)} \left| \frac{y-x}{|y-x|^{s+1}} - \frac{y-z}{|y-z|^{s+1}} \right| d\mu y \\ &+ \left| \int_{B(z,1) \setminus B(z,2d)} \frac{y-z}{|y-z|^{s+1}} d\mu y \right| \\ &\leq 2^s + 4^s + C2^{2s+1} + 2\tilde{C}_{s,\mu}^*(z), \end{aligned}$$

where the estimate for the first term is obvious, in the second we have used the fact that the domain of integration is contained in $B(x, 2) \setminus B(x, 1/2)$ and the third term is estimated by (2). If $d \geq 1/2$ or $\epsilon > 2d$, similar estimates give the result. $\square$

3. TANGENT MEASURES AND RIESZ s -TRANSFORMS

Let μ_i , $i = 1, 2, \dots$, and μ be locally finite Borel measures in $\mathbb{R}^n$. We say that measures (μ_i) converge weakly to μ if $\int \varphi d\mu_i \rightarrow \int \varphi d\mu$ for every continuous compactly supported function $\varphi : \mathbb{R}^n \rightarrow \mathbb{R}$. This is denoted by $\mu_i \rightarrow \mu$.

Let μ be a locally finite Borel measure in $\mathbb{R}^n$. For any $a \in \mathbb{R}^n$ and $r > 0$ let $T_{a,r}\mu$ be the image of μ under the map $T_{a,r}$, $T_{a,r}(x) = (x - a)/r$, that is,

$$T_{a,r}\mu(A) = \mu(rA + a).$$

We say that ν is a *tangent measure* of μ at a if ν is a locally finite nonzero Borel measure in $\mathbb{R}^n$ and if there are positive numbers $r_i \downarrow 0$ such that

$$r_i^{-s} T_{a,r_i} \mu \rightarrow \nu.$$

We denote the set of all tangent measures of μ at a by $\text{Tan}(\mu, a)$. The use of the particular normalization constants r_i^{-s} is motivated by the density assumptions we shall make on the measure μ below. Under them our definition corresponds to that of Preiss in [P].

Basic properties of tangent measures can be found in [P] and [M].

The following lemma tells about the usefulness of tangent measures in the investigation of the boundedness of maximal Riesz s -transforms.

Lemma 4. *Let μ be a locally finite Borel measure in $\mathbb{R}^n$. Suppose that $0 < \underline{D}^s(\mu, x) \leq \overline{D}^s(\mu, x) < \infty$ for μ almost all $x \in \text{spt } \mu$ and $\tilde{C}_{s,\mu}^* < \infty$ μ almost everywhere. Then at μ almost all points $a \in \mathbb{R}^n$ for every $\nu \in \text{Tan}(\mu, a)$ there exist constants $0 < c_1 \leq c_2 < \infty$ such that*

$$(3) \quad c_1 r^s \leq \nu B(x, r) \leq c_2 r^s$$

for all $x \in \text{spt } \nu$ and $0 < r < \infty$. Additionally, for μ almost all $a \in \mathbb{R}^n$ and for every $\nu \in \text{Tan}(\mu, a)$

$$(4) \quad \tilde{C}_{s,\nu}^* \text{ is bounded on } \mathbb{R}^n.$$

Proof. Let $\epsilon > 0$ and B be a set such that

$$B = \{x \in \text{spt } \mu : 0 < c' r^s \leq \mu B(x, r) \leq c'' r^s < \infty \\ \text{for all } 0 < r < r_0 \text{ and } \tilde{C}_{s,\mu}^*(x) < M\},$$

where c' , c'' , r_0 , and M are positive constants. The set B can be chosen so that $\mu(\mathbb{R}^n \setminus B) < \epsilon$. It suffices to consider the μ density points of the set B , i.e. those points $a \in B$ where

$$\lim_{r \downarrow 0} \frac{\mu(B(a, r) \setminus B)}{\mu B(a, r)} = 0.$$

At these points $\text{Tan}(\mu, a) = \text{Tan}(\mu \llcorner B, a)$, where $\mu \llcorner B$ is the restriction of the measure μ to the set B .

Let $a \in B$ be a μ -density point of the set B and $\nu = \lim_{i \rightarrow \infty} r_i^{-s} T_{a, r_i} \mu \in \text{Tan}(\mu, a)$. Now, for every $x \in \text{spt } \nu$ and $r > 0$, by the lower semicontinuity of the weak convergence

$$0 < \nu U(x, r) \leq \liminf_{i \rightarrow \infty} r_i^{-s} T_{a, r_i} \mu(B(x, r)) \\ = \liminf_{i \rightarrow \infty} r_i^{-s} \mu(B(r_i x + a, r r_i) \cap B)$$

holds for open balls $U(x, r)$. Thus for any $r > 0$ the set $B(r_i x + a, r r_i) \cap B \neq \emptyset$ for all sufficiently large i . Therefore we can choose a sequence of points $a_i \in B$ such that $x_i = (a_i - a)/r_i \rightarrow x$. Now for any $r < t$

$$\nu B(x, r) \leq \nu U(x, t) \leq \liminf_{i \rightarrow \infty} r_i^{-s} \mu B(r_i x + a, t r_i) \\ \leq \liminf_{i \rightarrow \infty} r_i^{-s} \mu B(a_i, 2 t r_i) \\ \leq \liminf_{i \rightarrow \infty} r_i^{-s} c'' (2 t r_i)^s = c_2 t^s.$$

As $t \downarrow r$ we get $\nu B(x, r) \leq c_2 r^s$. The lower bound can be found similarly by using the fact that

$$\nu B(x, r) \geq \limsup_{i \rightarrow \infty} r_i^{-s} \mu B(r_i x + a, r r_i)$$

when $x \in \text{spt } \nu$ and $r > 0$.

For the proof of the second statement let the set B be as before, $x \in \text{spt } \nu$, $\nu \in \text{Tan}(\mu, a)$, and $0 < r < R < \infty$. As before, we can choose a sequence of points $a_i \in B$ such that $x_i = (a_i - a)/r_i \rightarrow x$. If $\nu(\partial B(x, R)) = \nu(\partial B(x, r)) = 0$, we find that

$$\left| \int_{B(x, R) \setminus B(x, r)} \frac{y - x}{|y - x|^{s+1}} d\nu y \right| = \lim_{i \rightarrow \infty} \left| \int_{B(x_i, R) \setminus B(x_i, r)} \frac{y - x_i}{|y - x_i|^{s+1}} d\nu y \right| \\ = \lim_{i \rightarrow \infty} \left| r_i^{-s} \int_{B(x_i, R) \setminus B(x_i, r)} \frac{y - x_i}{|y - x_i|^{s+1}} dT_{a, r_i} \mu y \right| \\ = \lim_{i \rightarrow \infty} \left| \int_{B(r_i x_i + a, R r_i) \setminus B(r_i x_i + a, r r_i)} \frac{y - (r_i x_i + a)}{|y - (r_i x_i + a)|^{s+1}} d\mu y \right| \\ = \lim_{i \rightarrow \infty} \left| \int_{B(a_i, R r_i) \setminus B(a_i, r r_i)} \frac{y - a_i}{|y - a_i|^{s+1}} d\mu y \right| \leq 2M.$$

By approximation this holds for all $0 < r < R < \infty$. The conditions of Lemma 3 hold for the measure ν and thus $\tilde{C}_{s,\nu}^*$ is bounded on $\mathbb{R}^n$. $\square$

Theorem 5. *Let μ be a locally finite nonzero Borel measure in $\mathbb{R}^n$. Suppose that $0 < \underline{D}^s(\mu, x) \leq \overline{D}^s(\mu, x) < \infty$ for μ almost all $x \in \text{spt } \mu$. If $\tilde{C}_{s,\mu}^* < \infty$ μ almost everywhere, then s is an integer.*

Proof. If the theorem is false, $\tilde{C}_{s,\mu}^* < \infty$ μ almost everywhere for some measure μ satisfying the assumptions of the theorem in some $\mathbb{R}^n$ for a non-integral s . Let n be the smallest possible dimension where this can happen. We want to show that this happens also in $\mathbb{R}^{n-1}$.

Let μ be a measure in $\mathbb{R}^n$ satisfying the above assumptions and $a \in \mathbb{R}^n$. We can find a tangent measure $\nu \in \text{Tan}(\mu, a)$ such that the conditions (3) and (4) of Lemma 4 hold. Because $s < n$ the support of ν cannot be the whole space $\mathbb{R}^n$. This implies that there exist points $b \notin \text{spt } \nu$ and $c \in \text{spt } \nu$ and a number $r > 0$ such that

$$\begin{aligned} U(b, r) \cap \text{spt } \nu &= \emptyset \quad \text{and} \\ c &\in \partial B(b, r) \cap \text{spt } \nu. \end{aligned}$$

Without loss of generality we may assume that $c = 0$ and b is on the negative x_1 -axis. Because $0 \in \text{spt } \nu$, the condition (3) holds for the measure ν at the point 0 and therefore both (3) and (4) hold for every tangent measure $\sigma \in \text{Tan}(\nu, 0)$ as one can see from the proof of Lemma 4. Now $\text{spt } \sigma \subset H = \{x \in \mathbb{R}^n : x_1 \geq 0\}$.

Because n was chosen to be minimal, $\text{spt } \sigma \not\subset \partial H$. Denote $S_\delta = \{x \in \mathbb{R}^n : x_1 \geq \delta|x|\}$. Now there exists $\delta < 1$, and a sequence $x_i \in S_\delta \cap \text{spt } \sigma \setminus \{0\}$ such that $x_i \rightarrow 0$. Otherwise we should have, for some $\lambda = \lim r_i^{-s} T_{0,r_i} \sigma$ with $r_i \downarrow 0$ and for $M > 0$, $\delta > 0$,

$$\begin{aligned} \lambda \left(\text{int}(S_\delta \cap U(0, M)) \right) &\leq \liminf_{i \rightarrow \infty} r_i^{-s} T_{0,r_i} \sigma \left(\text{int}(S_\delta \cap U(0, M)) \right) \\ &\leq \liminf_{i \rightarrow \infty} r_i^{-s} \sigma(S_\delta \cap B(r_i M)) = 0. \end{aligned}$$

Thus $\text{spt } \lambda \subset \partial H$ which contradicts the minimality of n .

We may assume $|x_1| > |x_2| > \dots$ and by passing to a subsequence that the balls $B_i = B(x_i, \delta|x_i|/2)$ are disjoint and contained in the ball $B(0, 1)$. Then for any k

$$\begin{aligned} \int_{B(0,1) \setminus B(0,|x_k|/2)} \frac{y_1}{|y|^{s+1}} d\sigma y &\geq \sum_{i=1}^k \int_{B_i} \frac{y_1}{|y|^{s+1}} d\sigma y \\ &\geq \sum_{i=1}^k \frac{\delta}{2^{s+2}|x_i|^s} \sigma B_i \\ &\geq \sum_{i=1}^k c_1 \frac{\delta}{2^{s+2}|x_i|^s} \left(\frac{\delta|x_i|}{2} \right)^s \\ &= c_1 2^{-2s-2} \delta^{s+1} k \end{aligned}$$

and so

$$\begin{aligned} \left| \int_{B(0,1) \setminus B(0,|x_k|/2)} \frac{y}{|y|^{s+1}} d\sigma y \right| &\geq \left| \int_{B(0,1) \setminus B(0,|x_k|/2)} \frac{y_1}{|y|^{s+1}} d\sigma y \right| \\ &\geq c_1 2^{-2s-2} \delta^{s+1} k \xrightarrow{k \rightarrow \infty} \infty. \end{aligned}$$

Thus

$$\sup_{\epsilon > 0} \left| \int_{B(0,1) \setminus B(0,\epsilon)} \frac{y}{|y|^{s+1}} d\sigma y \right| = \infty.$$

This contradicts (4) in Lemma 4. Assumption that there would be a minimal n is false and the result follows.

Note that due to Lemma 2 the almost everywhere finiteness of $\tilde{\mathcal{C}}_{s,\mu}^*$ can be replaced by the essential boundedness of $\mathcal{C}_{s,\mu}$ if μ also satisfies the conditions of Lemma 2. $\square$

4. L^2 -BOUNDEDNESS OF RIESZ s -TRANSFORMS

Next we shall study Riesz s -transforms as singular integral operators. Let μ be a positive Radon measure in $\mathbb{R}^n$ for which there exist constants $0 < c_1 \leq c_2 < \infty$ such that

$$(5) \quad c_1 r^s \leq \mu B(x, r) \leq c_2 r^s$$

for all $x \in \text{spt } \mu$ and $0 < r < \text{diam}(\text{spt } \mu)$. We define a singular integral operator $\mathcal{C}_{s,\mu}$ by

$$\langle \mathcal{C}_{s,\mu} f, g \rangle = \frac{1}{2} \iint \frac{y-x}{|y-x|^{s+1}} (f(y)g(x) - g(y)f(x)) d\mu y d\mu x$$

for all $f, g \in L^2(\mu)$ whose supports are separated by a positive distance (see [C] for more details). Saying that the operator $\mathcal{C}_{s,\mu}$ is bounded on L^2 means that there exists a constant $C < \infty$ such that

$$\|\mathcal{C}_{s,\mu} f\|_2 \leq C \|f\|_2$$

for $f \in L^2(\mu)$. We shall show that if there exists a nonzero measure μ such that it satisfies (5) and the singular integral operator $\mathcal{C}_{s,\mu}$ is bounded on L^2 , then s is an integer.

Theorem 6. *Let μ be a nonzero Radon measure in $\mathbb{R}^n$ such that for $x \in \text{spt } \mu$ and $0 < r < \text{diam}(\text{spt } \mu)$*

$$0 < c_1 r^s \leq \mu B(x, r) \leq c_2 r^s < \infty.$$

If $\mathcal{C}_{s,\mu}$ is bounded on L^2 , then s is an integer.

Proof. We first note that if the singular integral operator is bounded on L^2 , then also the maximal truncated operator $\tilde{\mathcal{C}}_{s,\mu}^*$ is bounded on L^2 , see [C] or [J]. Then also $\tilde{\mathcal{C}}_{s,\mu}^* < \infty$ μ almost everywhere and we can use Theorem 5 to get the result. $\square$

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