ON THE PLURICANONICAL MAP OF THREEFOLDS OF GENERAL TYPE

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(Communicated by Eric M. Friedlander)

ABSTRACT. Let X be a smooth minimal threefold of general type and let n be an integer > 1 . Assume that the image of the pluricanonical map Φ_n of X is a curve. Then a simple computation shows that n is necessarily 2 or 3. When $n = 2$ with a numerical condition or when $n = 3$, we obtain two inequalities $\chi(\mathcal{O}_X) \leq \min\{-1, 2 - 2q_1\}$ and $q_1 \leq \frac{3}{14}K_X^3 + 1$, where q_1 is the irregularity of X and $\chi(\mathcal{O}_X)$ is the Euler characteristic of X .

Throughout this paper, we are working over the complex number field $\mathbf{C}$.

In this paper, we have studied the case that the image C of the pluricanonical map Φ_n is a curve for an integer $n > 1$. In this case, a simple calculation shows $n = 2$ or 3. Resolve the base locus of Φ_n . Then we have two terms ‘ b ’ and ‘ c ’ explained just below Proposition 1. When $n = 2$ with additional numerical conditions or when $n = 3$, we may have some information about Φ_n from these two terms, which are explained in Corollary of Proposition 2 and Proposition 3. Using this information, we obtain two inequalities $\chi(\mathcal{O}_X) \leq \min\{-1, 2 - 2q_1\}$ and $q_1 \leq \frac{3}{14}K_X^3 + 1$, which are given in Theorems 2 and 3.

Now, let’s set up our notations. Let X be a smooth projective variety and let D be a divisor on X . Denote by K_X the canonical divisor of X . Denote by Φ_n the rational map associated to the complete linear system $|nK_X|$. Denote by $h^i(X, \mathcal{O}_X(D))$ the dimension of $H^i(X, \mathcal{O}_X(D))$. Let $\text{Bs}|D|$ mean the base locus of $|D|$. Let’s denote the genus of X by $p_g(X)$ and $h^i(X, \mathcal{O}_X)$ by $q_i(X)$. (or simply p_g and q_i unless there is any confusion.) Denote by $\sim$ the linear equivalence and by $\equiv$ the numerical equivalence. For a real number r , $[r]$ means the greatest among the integers less than or equal to r .

Theorem 1 (Kawamata-Viehweg vanishing theorem). *Let X be a nonsingular projective variety. If D is a nef and big divisor on X , then $H^i(X, \mathcal{O}_X(K_X + D)) = 0$ for all $i > 0$.*

For a reference, see KMM [5].

Lemma. *Let X be a smooth projective threefold, and let D be a divisor on X . Then we have the following:*

Received by the editors June 12, 1995.

1991 *Mathematics Subject Classification.* Primary 14E05, 14J30.

Key words and phrases. Threefold of general type, pluricanonical map, fiber space, Euler characteristic.

This paper is supported by KOSEF and Dae-Yang Foundation.

- (a) $\chi(\mathcal{O}_X(D)) = D^3/6 - K_X \cdot D^2/4 + D \cdot (K_X^2 + c_2)/12 + \chi(\mathcal{O}_X)$, where c_2 is the second Chern class of X . Moreover $\chi(\mathcal{O}_X) = -c_2 \cdot K_X/24$.
- (b) $K_X \cdot D^2$ is even.
- (c) $p_n \stackrel{\text{def}}{=} h^0(X, \mathcal{O}_X(nK_X)) = \frac{n(n-1)(2n-1)}{12} K_X^3 + (1-2n)\chi(\mathcal{O}_X)$ for $n \geq 2$ and $\chi(\mathcal{O}_X) < 0$ when K_X is nef and big.

Proof. (a) comes from the Riemann-Roch theorem.

(b) comes from the following:

$$\chi(\mathcal{O}_X(D)) + \chi(\mathcal{O}_X(-D)) = -K_X \cdot D^2/2 + 2\chi(\mathcal{O}_X) \in \mathbb{Z}.$$

(c) comes from (a) and the Kawamata-Viehweg vanishing theorem. We have $\chi(\mathcal{O}_X) < 0$ since $-c_2 \cdot K_X \leq -K_X^3/3$. (See Miyaoka [8].) $\square$

Proposition 1. *Let C be a nondegenerate curve of degree a in $\mathbb{P}^n$. Then we have*

- (a) *If $n \leq a < 2n$, then $p_g(C) \leq a - n$.*
- (b) *If $2n \leq a$, then $p_g(C) \leq \frac{m(m-1)}{2}(n-1) + mr$, where $m = \left\lfloor \frac{a-1}{n-1} \right\rfloor$ and $a-1 = m(n-1) + r$.*

Proof. See Griffiths & Harris [2], p. 253. $\square$

We are going to use the following notations in the rest of this paper.

Let X be a smooth projective threefold of general type with K_X nef. Let n be a positive integer ≥ 2 .

Suppose that the dimension of the image C of Φ_n is 1.

Let $|nK_X| = |M| + Z$, where $|M|$ and Z are the moving part and fixed part of $|nK_X|$ respectively. Let $f: X' \rightarrow X$ be the resolution of the base locus of Φ_n if $\text{Bs}|nK_X| \neq \emptyset$, i.e., f is a succession of blow-ups along nonsingular centers of codim ≥ 2 in the base locus such that $g = \Phi_n \circ f$ is a morphism. Let $g = k \circ h$ be the Stein factorization. Observe that C' is normal and hence smooth.

$$\begin{array}{ccc} X' & \xrightarrow{h} & C' \\ f \downarrow & & \downarrow k \\ X & \xrightarrow{\Phi_n} & C \end{array}$$

Let a be $\deg C$ in $\mathbb{P}^{p_n-1}$, and let $b = \deg k$. Recall that C is a nondegenerate curve in $\mathbb{P}^{p_n-1}$. Let S be the general fiber of h .

We have $K_{X'} = f^*(K_X) + E'$ and $|f^*nK_X| = |M'| + Z'$, where E' is the ramification divisor of f supported on the exceptional locus of f and Z' is the fixed part of $|f^*nK_X|$. We have $M' \equiv abS$. Since f^*K_X is nef, $f^*K_X^2 \cdot Z' \geq 0$. Hence

$$nK_X^3 = nf^*K_X^3 = (abS + Z')f^*K_X^2 \geq abf^*K_X^2 \cdot S.$$

Let $c = f^*K_X^2 \cdot S$. Since f^*K_X is nef and big, and S is nef and not numerically equivalent to 0, we have $f^*K_X^2 \cdot S \geq 1$. Hence

$$(1) \quad \frac{nK_X^3}{bc} \geq a \geq p_n - 1.$$

From the inequality (1) and Lemma, the image of Φ_n can be a curve only when n is 2 or 3. Moreover, we have $1 \leq bc \leq 3$ and in particular, $bc = 1$ when $n = 3$.

Proposition 2 (cf. Matsuki [7]). *If $\dim \operatorname{Im} \Phi_n = 1$, then S is a surface of general type and S' has $K_{S'}^2 = f^*K_X^2 \cdot S$, where $\pi : S \rightarrow S'$ is a morphism of S to its minimal surface S' .*

Proof. The easy addition formula ' $\kappa(X) \leq \kappa(S) + \dim C$ ' implies that S is the surface of general type, where $\kappa(X)$ means the Kodaira dimension of X . (For a reference about easy addition formula, see Ueno [9].) We have that $nK_X \equiv abD + Z$ and $f^*abD \equiv abS + E$, where $D = f_*S$ and E is an effective divisor supported on the exceptional locus of f . So,

$$nf^*K_X^2 \cdot S = nK_X^2 \cdot D = (abD + Z) \cdot K_X \cdot D \geq abK_X \cdot D^2.$$

Since D^2 is an effective 1-cycle and K_X is nef, $K_X \cdot D^2 \geq 0$. If $K_X \cdot D^2 \neq 0$, then $nf^*K_X^2 \cdot S \geq 2a \geq 2(p_n - 1)$ since $K_X \cdot D^2$ is even. This inequality holds true only when $n = 2$, $K_X^3 = 2$, $\chi(\mathcal{O}_X) = -1$ and $f^*K_X^2 \cdot S = 3$. (Recall that $2 \leq n \leq 3$ and $1 \leq c \leq 3$.) But it does not satisfy the inequality (1). Hence $K_X \cdot D^2 = 0$. So

$$\begin{aligned} 0 &= (abD)^2 \cdot K_X = f^*abD \cdot f^*abD \cdot f^*K_X \\ &= (abS + E) \cdot f^*abD \cdot f^*K_X \\ &= abS \cdot f^*abD \cdot f^*K_X \\ &= abS \cdot (abS + E) \cdot f^*K_X \\ &= abS \cdot E \cdot f^*K_X. \end{aligned}$$

Hence we have $S \cdot E \cdot f^*K_X = 0$. Let $\{E_i\}$ be the irreducible components of E . Clearly $S \cdot E_i \cdot f^*K_X = 0$ for each i . By the way of taking f , $\operatorname{supp}(E) = \operatorname{supp}(E')$. Hence we have $S \cdot E' \cdot f^*K_X = 0$, since $S \cdot E_i \cdot f^*K_X = 0$ for each i .

Applying the Hodge index theorem to S , we have that $(E'|_S)^2 = S \cdot E'^2 \leq 0$ since $(f^*K_X|_S)^2 = f^*K_X^2 \cdot S \geq 1$. We have $f^*K_X|_S + E'|_S \sim K_S \sim \pi^*K_{S'} + L$, where L is an effective divisor supported on the exceptional locus of π . Hence the uniqueness of the Zariski decomposition implies that $f^*K_X|_S \sim \pi^*K_{S'}$. $\square$

Corollary. *If $f^*K_X^2 \cdot S = 1$, then the minimal surface S' of S has $K_{S'}^2 = 1$, $q(S) = 0$ and $1 \leq p_g(S) \leq 2$.*

Proof. $K_{S'}^2 = f^*K_X|_S^2 = 1$ by Proposition 2. Since $K_{S'}^2 = 1$, we have that $q(S) = 0$ and $p_g(S) \leq 2$. (For a reference, see Bombieri [1], p. 212.) Consider the exact sequence

$$0 \rightarrow \mathcal{O}_{X'}(K_{X'}) \rightarrow \mathcal{O}_{X'}(K_{X'} + M') \rightarrow \bigoplus^{ab} \mathcal{O}_S(K_S) \rightarrow 0.$$

The above exact sequence shows that $p_g(S) \neq 0$ since M' is not fixed. $\square$

We have two facts from the condition $bc = 1$. The first one is that C' is birational to the image C of the pluricanonical map Φ_n . So we may assume that C is smooth since the terms we are interested in are birational invariants. The second one is that a general fiber of g is a surface of general type with its irregularity 0 from Corollary of Proposition 2.

Proposition 3. *Suppose that $bc = 1$. Then we have that $1 \leq p_g(S) \leq 2$ and $\chi(\mathcal{O}_X) \leq (p_g(S) + 1)(1 - q_1)$.*

Proof. We have the fiber space $g : X' \rightarrow C$ with connected fiber since $bc = 1$. By Corollary of Proposition 2, $q(S) = 0$ and $1 \leq p_g(S) \leq 2$. We have $R^1 g_* K_{X'} = 0$ since $q(S) = h^1(S, \mathcal{O}_S(K_S)) = 0$. By spectral sequence, we have that

$$\begin{aligned} p_g &= h^0(X, \mathcal{O}_X(K_X)) = h^0(C, g_* K_{X'}), \\ q_2 &= h^1(X, \mathcal{O}_X(K_X)) = h^1(C, g_* K_{X'}), \\ q_1 &= h^2(X, \mathcal{O}_X(K_X)) = h^0(C, R^2 g_* K_{X'}). \end{aligned}$$

Since $R^2 g_* K_{X'} = K_C$, $q_1 = p_g(C)$. (For a reference, see Kollár [6].) It is known that $g_* K_{X'/C} \stackrel{\text{def}}{=} g_*(K_{X'} \otimes g^* K_C^{-1})$ is semipositive and locally free of rank $p_g(S)$. So $\deg g_* K_{X'/C} \geq 0$. (See Kawamata [4].)

$$\begin{aligned} h^0(C, g_* K_{X'}) - h^1(C, g_* K_{X'}) &= \deg g_* K_{X'} + p_g(S)(1 - p_g(C)) \\ &= \deg g_* K_{X'/C} + p_g(S)(p_g(C) - 1) \\ &\geq p_g(S)(p_g(C) - 1). \end{aligned}$$

So $p_g - q_2 \geq p_g(S)(q_1 - 1)$. Hence $-\chi(\mathcal{O}_X) = p_g - q_2 + q_1 - 1 \geq (p_g(S) + 1)(q_1 - 1)$. Now we have $\chi(\mathcal{O}_X) \leq (p_g(S) + 1)(1 - q_1)$. $\square$

Theorem 2. *If $\dim \operatorname{Im} \Phi_3 = 1$, then the following hold:*

- (a) $-\frac{1}{10}K_X^3 - \frac{1}{5} \leq \chi(\mathcal{O}_X) \leq \min\{-1, 2 - 2q_1\}$. Moreover, $K_X^3 \geq 8$.
- (b) $q_1 \leq \frac{1}{22}K_X^3 + 1$.

Proof. The fact $bc = 1$ is given just above Proposition 2. If $a \geq 2(p_3 - 1)$, then the inequality (1) implies $3K_X^3/2 \geq p_3 - 1$. But it is impossible since $p_3 = \frac{5}{2}K_X^3 - 5\chi(\mathcal{O}_X)$. So $bc = 1$ and $a < 2(p_3 - 1)$. Hence Lemma and Proposition 3 imply $\chi(\mathcal{O}_X) \leq \min\{-1, 2 - 2q_1\}$. From the inequality (1), we have $-\frac{1}{10}K_X^3 - \frac{1}{5} \leq \chi(\mathcal{O}_X)$. Moreover, since $\chi(\mathcal{O}_X) \leq -1$, $K_X^3 \geq 8$.

For (b), by Proposition 1, $p_g(C) \leq a - (p_3 - 1)$ since the degree a of C is less than $2(p_3 - 1)$. Recall that we have $q_1 = p_g(C)$ in the proof of Proposition 3. We already know $a \leq 3K_X^3$ from (1). Hence

$$q_1 = p_g(C) \leq a - (p_3 - 1) \leq 3K_X^3 - p_3 + 1 \leq 1/2K_X^3 + 5\chi(\mathcal{O}_X) + 1.$$

From (a), $q_1 \leq 1/2K_X^3 + 5(2 - 2q_1) + 1$. Hence we have $q_1 \leq \frac{1}{22}K_X^3 + 1$. $\square$

Theorem 3. *Suppose that $\dim \operatorname{Im} \Phi_2 = 1$. If $bc = 1$, then we have the following:*

- (a) $\chi(\mathcal{O}_X) \leq \min\{-1, 2 - 2q_1\}$.
- (b) $q_1 \leq \frac{3}{14}K_X^3 + 1$.

Proof. Lemma and Proposition 3 imply (a) clearly.

For (b), we already know $p_2 - 1 \leq a \leq 2K_X^3$ from (1). Hence it is enough to consider the following two cases:

Case 1. $p_2 - 1 \leq a < 2(p_2 - 1)$.

Since $p_2 - 1 \leq a < 2(p_2 - 1)$, Proposition 1 implies that $p_g(C) \leq a - (p_2 - 1)$. Since $bc = 1$, the proof of Proposition 3 implies that $q_1 = p_g(C)$ and $\chi(\mathcal{O}_X) \leq 2 - 2q_1$.

$$\begin{aligned} q_1 = p_g(C) &\leq a - (p_2 - 1) \\ &\leq 2K_X^3 - p_2 + 1 \\ &= \frac{3}{2}K_X^3 + 3\chi(\mathcal{O}_X) + 1 \\ &\leq \frac{3}{2}K_X^3 + 3(2 - 2q_1) + 1. \end{aligned}$$

Hence we have $q_1 \leq \frac{3}{14}K_X^3 + 1$.

Case 2. $2(p_2 - 1) \leq a \leq 2K_X^3$.

Since $2(p_2 - 1) \leq a \leq 2K_X^3$, Proposition 1 implies that

$$p_g(C) \leq \frac{m(m-1)}{2}(p_2 - 2) + mr,$$

where $m = \left\lceil \frac{a-1}{p_2-2} \right\rceil$ and $a-1 = m(p_2-2) + r$.

So, let's compute m . Since $2(p_2 - 1) \leq a \leq 2K_X^3$, we have that

$$\frac{2(p_2 - 1) - 1}{p_2 - 2} \leq \frac{a - 1}{p_2 - 2} \leq \frac{2K_X^3 - 1}{p_2 - 2}.$$

If we modify the above inequalities, we have that

$$2 + \frac{1}{K_X^3/2 - 3\chi(\mathcal{O}_X) - 2} \leq \frac{a-1}{p_2-2} \leq 4 + \frac{12\chi(\mathcal{O}_X) + 7}{K_X^3/2 - 3\chi(\mathcal{O}_X) - 2}.$$

Since $\chi(\mathcal{O}_X) < 0$, we have that

$$\frac{1}{K_X^3/2 - 3\chi(\mathcal{O}_X) - 2} > 0 \text{ and } \frac{12\chi(\mathcal{O}_X) + 7}{K_X^3/2 - 3\chi(\mathcal{O}_X) - 2} < 0.$$

Hence we have $m = \left\lceil \frac{a-1}{p_2-2} \right\rceil = 2$ or 3 .

When $m = 2$, we have that

$$\begin{aligned} q_1 = p_g(C) &\leq \frac{2 \cdot 1}{2}(p_2 - 2) + 2r \\ &\leq (p_2 - 2) + 2(a - 1 - 2(p_2 - 2)) \\ &\leq 2a - 3p_2 + 4 \\ &\leq 4K_X^3 - 3\left(\frac{K_X^3}{2} - 3\chi(\mathcal{O}_X)\right) + 4 \\ &\leq \frac{5}{2}K_X^3 + 9\chi(\mathcal{O}_X) + 4 \\ &\leq \frac{5}{2}K_X^3 + 9(2 - 2q_1) + 4. \end{aligned}$$

Hence $q_1 \leq \frac{5}{38}K_X^3 + \frac{22}{19}$.

When $m = 3$, similarly, we have that $q_1 \leq \frac{3}{37}K_X^3 + \frac{45}{37}$.

Therefore, combining all inequalities about q_1 , we have $q_1 \leq \frac{3}{14}K_X^3 + 1$. $\square$

Remark. When $\dim \operatorname{Im} \Phi_2 = 1$, we assume the condition $bc = 1$. At this moment, we don't have a necessary and sufficient condition to guarantee $bc = 1$. But we have some cases which show $bc = 1$.

Proposition 4. *Suppose that $\dim \operatorname{Im} \Phi_2 = 1$. If $K_X^3 < p_2 - 1$, then we have that*

- (a) $bc = 1$,
- (b) $-K_X^3/2 - 1/3 \leq \chi(\mathcal{O}_X) < -K_X^3/6 - 1/3$.

Proof. Since $K_X^3 < p_2 - 1 \leq a$, we have $K_X^3 < p_2 - 1 \leq a \leq \frac{2K_X^3}{bc}$ from (1). Hence bc must be 1.

From (1), we have $-K_X^3/2 - 1/3 \leq \chi(\mathcal{O}_X)$. Since $K_X^3 < p_2 - 1$, we have $\chi(\mathcal{O}_X) < -K_X^3/6 - 1/3$. Combining these two inequalities, we have

$$-K_X^3/2 - 1/3 \leq \chi(\mathcal{O}_X) < -K_X^3/6 - 1/3.$$

□

Proposition 5. *Suppose that $\dim \operatorname{Im} \Phi_2 = 1$. If $2(p_2 - 1) \leq a$, then we have*

- (a) $bc = 1$,
- (b) $-K_X^3/6 - 1/3 \leq \chi(\mathcal{O}_X) \leq -1$.

Proof. Since $2(p_2 - 1) \leq a$, we have $2(p_2 - 1) \leq a \leq 2K_X^3/bc$ from (1). If $bc \geq 2$, then we have $2(p_2 - 1) \leq K_X^3$. But, since $2(p_2 - 1) = K_X^3 - 6\chi(\mathcal{O}_X) - 2 > K_X^3$, it is impossible. Hence $bc = 1$.

From $2(p_2 - 1) \leq a \leq 2K_X^3$, we have $-K_X^3/6 - 1/3 \leq \chi(\mathcal{O}_X)$. □

Remark. When $\dim \operatorname{Im} \Phi_2 = 1$, from (1), we may have the following three cases: $a \leq 2K_X^3 < 2(p_2 - 1)$, $a \leq 2(p_2 - 1) \leq 2K_X^3$, and $2(p_2 - 1) \leq a \leq 2K_X^3$. The case we didn't cover here is the second one $p_2 - 1 \leq a \leq 2(p_2 - 1) \leq 2K_X^3$.

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