THE OPTIMALITY OF JAMES'S DISTORTION THEOREMS

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ABSTRACT. A renorming of ℓ_1 , explored here in detail, shows that the copies of ℓ_1 produced in the proof of the Kadec-Pelczyński theorem inside nonreflexive subspaces of $L_1[0, 1]$ cannot be produced inside general nonreflexive spaces that contain copies of ℓ_1 . Put differently, James's distortion theorem producing one-plus-epsilon-isomorphic copies of ℓ_1 inside any isomorphic copy of ℓ_1 is, in a certain sense, optimal. A similar renorming of c_0 shows that James's distortion theorem for c_0 is likewise optimal.

James's distortion theorems for ℓ_1 , the space of absolutely summable sequences of scalars, and c_0 , the space of null sequences of scalars, are well-known [J]. The former states that, whenever a Banach space contains a subspace isomorphic to ℓ_1 , the Banach space contains subspaces that are almost isometric to ℓ_1 . Several of the authors of this article, individually and in concert, have tried to use this feature of ℓ_1 to determine if all (equivalent) renormings of ℓ_1 fail to have the fixed point property for nonexpansive mappings (the FPP); i.e. if, in any renorming of ℓ_1 , there exist a nonempty, closed, bounded and convex subset C and a nonexpansive self-map T of C without a fixed point. The basis of these attempts was to use the fact that ℓ_1 in its usual norm fails to have the fixed point property and, since each renorming of ℓ_1 contains subspaces almost isometric to ℓ_1 , a perturbation of the usual example would hopefully produce a nonexpansive self-map of a nonempty, closed, bounded, convex set in any renorming of ℓ_1 . Similar attempts in c_0 were also made. What appeared to be needed in these attempts were strengthened versions of James's distortion theorems.

To be specific, James's theorem for ℓ_1 states that if a Banach space X with norm $\|\cdot\|$ contains an isomorphic copy of ℓ_1 , then, for each $\epsilon > 0$, there exists a sequence (x_k) in the unit sphere of X such that $(1 - \epsilon) \sum_{k=1}^{\infty} |t_k| \leq \|\sum_{k=1}^{\infty} t_k x_k\| \leq \sum_{k=1}^{\infty} |t_k|$ for all $(t_k) \in \ell_1$. The proof of the theorem shows even more than the statement indicates. The sequence (x_k) may be chosen to have the additional property that, if (ϵ_n) is a sequence of positive numbers decreasing to 0, then for each n , $(1 - \epsilon_n) \sum_{k=n}^{\infty} |t_k| \leq \|\sum_{k=n}^{\infty} t_k x_k\| \leq \sum_{k=n}^{\infty} |t_k|$, for all $(t_k) \in \ell_1$. That is, for each $\delta > 0$, by ignoring a finite number of terms at the beginning of the sequence (x_k) , one obtains copies of ℓ_1 which are $(1 + \delta)$ -isomorphic to ℓ_1 . This

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leads one to ask if James's distortion theorem can be strengthened in the following sense:

Question. *If X is a Banach space that contains an isomorphic copy of ℓ_1 and (ϵ_n) is a sequence of positive numbers that decreases to 0, does there exist a sequence (x_k) in the unit sphere of X such that $\sum_{k=1}^{\infty} (1 - \epsilon_k) |t_k| \leq \|\sum_{k=1}^{\infty} t_k x_k\| \leq \sum_{k=1}^{\infty} |t_k|$ for all $(t_k) \in \ell_1$?*

The closed linear span of such a sequence (x_n) in the above question is called an *asymptotically isometric copy* of ℓ_1 . As noted in [DL], the proof of the Kadec-Pełczyński theorem [KP] shows that nonreflexive subspaces of $L_1[0, 1]$ contain such “good” copies of ℓ_1 and, in this case, there exist nonexpansive self-maps on closed, bounded and convex sets without fixed points. (This provides a converse to a theorem of Maurey [M] that every reflexive subspace of $L_1[0, 1]$ has the FPP.) Thus, if every renorming of ℓ_1 were to contain an asymptotically isometric copy of ℓ_1 , then every renorming of ℓ_1 would fail the fixed point property. One purpose of this article is to present a renorming of ℓ_1 which contains no asymptotically isometric copy of ℓ_1 . Thus James's distortion theorem for ℓ_1 is, in this sense, optimal and the question of whether ℓ_1 can be given an equivalent norm with the fixed point property remains open. Using the predual of this renorming of ℓ_1 , it will be seen that James's distortion theorem for c_0 is similarly optimal and the question as to whether c_0 can be given an equivalent norm with the fixed point property likewise remains open.

Recent papers ([CDL, DLT]) have extended the classes of spaces known to contain asymptotically isometric copies of ℓ_1 . In a related paper, Smyth [S] showed that the dual of every space $C(\Omega)$, where Ω is an infinite compact Hausdorff space, fails the weak-star fixed point property with an affine contraction.

In the ensuing discussion, $\mathbb{K}$ will denote the scalar field (the real or the complex numbers) and $\mathbb{N}$ will denote the positive integers. The Banach space ℓ_1 is as usual the space of absolutely summable scalar sequences with its usual norm $\|x\|_1 := \sum_{n=1}^{\infty} |\xi_n|$, for all $x = (\xi_n) \in \ell_1$. More generally, for $p \geq 1$, the Banach space of p -summable sequences of scalars is denoted by ℓ_p and is normed by $\|x\|_p := (\sum_{n=1}^{\infty} |\xi_n|^p)^{1/p}$ for all $x = (\xi_n) \in \ell_p$. The sequence (e_n) will always denote the canonical unit vector basis in ℓ_p . Recall that $\|x\|_p \leq \|x\|_1$ for $p \geq 1$ and $x \in \ell_1$.

The space to be defined is, on the surface, quite simple. It is a countable sum of $\mathbb{K}$'s and is akin to the classical ℓ_p -spaces. There are two significant features to notice: the varying values of the exponents (similar to spaces of Nakano) and the placement of the parentheses in defining the norm. Fix a sequence $p = (p_n)$ of real numbers in $(1, \infty)$ converging to 1. Then the space we wish to define is:

$$\mathbb{K} \oplus_{p_1} (\mathbb{K} \oplus_{p_2} (\mathbb{K} \oplus_{p_3} (\mathbb{K} \oplus_{p_4} \dots))).$$

Let $X := \mathbb{K}^{\mathbb{N}}$. For $x = (\xi_n) \in X$, define:

$$\nu_1(p, x) := |\xi_1| ,$$

$$\nu_2(p, x) := (|\xi_1|^{p_1} + |\xi_2|^{p_1})^{1/p_1} ,$$

$$\nu_3(p, x) := \left(|\xi_1|^{p_1} + (|\xi_2|^{p_2} + |\xi_3|^{p_2})^{p_1/p_2} \right)^{1/p_1} ,$$

$$\nu_4(p, x) := \left(|\xi_1|^{p_1} + \left(|\xi_2|^{p_2} + (|\xi_3|^{p_3} + |\xi_4|^{p_3})^{p_2/p_3} \right)^{p_1/p_2} \right)^{1/p_1}.$$

To proceed further with this inductive construction, some notation is useful. Define the shift operator $S : X \rightarrow X$ by $Sz := (z_2, z_3, \dots, z_k, \dots)$ for all $z \in X$. For p and x as above define, for each $n \in \mathbb{N}$,

$$\nu_{n+1}(p, x) := (|\xi_1|^{p_1} + \nu_n(Sp, Sx)^{p_1})^{1/p_1}.$$

Each $\nu_n(p, \cdot)$ is a seminorm on X and, for each $x \in X$, the sequence $(\nu_n(p, x))_{n=1}^\infty$ increases to a limit $\nu_p(x)$. Clearly, for all $x \in X$, $\nu_n(p, x) \leq |\xi_1| + \dots + |\xi_n|$ for every n . Thus $\nu_p(x) \leq \|x\|_1$ for each $x \in \ell_1$.

In seeking lower estimates for $\nu_p(x)$, first note that since all two-dimensional normed linear spaces are equivalent, two-dimensional ℓ_q^2 is equivalent to ℓ_1^2 and in fact, for $q \geq 1$,

$$\|(\xi_1, \xi_2)\|_q \geq 2^{-1+1/q} \|(\xi_1, \xi_2)\|_1.$$

Then, with $K_j = 2^{-1+1/p_j}$,

$$\begin{aligned} \nu_n(p, x) &= (|\xi_1|^{p_1} + \nu_{n-1}(Sp, Sx)^{p_1})^{1/p_1} \\ &\geq K_1 (|\xi_1| + \nu_{n-1}(Sp, Sx)) \\ &= K_1 \left(|\xi_1| + (|\xi_2|^{p_2} + \nu_{n-2}(S^2p, S^2x)^{p_2})^{1/p_2} \right) \\ &\geq K_1 (|\xi_1| + K_2 (|\xi_2| + \nu_{n-2}(S^2p, S^2x))) \\ &\geq K_1 K_2 (|\xi_1| + |\xi_2| + \nu_{n-2}(S^2p, S^2x)) \\ &\vdots \\ &\geq K_1 K_2 \cdots K_n \sum_{j=1}^n |\xi_j|. \end{aligned}$$

Specializing to the sequence $p = (p_j)$ where $p_j = \frac{2^j}{2^j - 1}$ yields:

$$\frac{1}{2} \|x\|_1 \leq \nu_p(x) \leq \|x\|_1 \quad \text{for all } x \in \ell_1.$$

Thus, for this specific choice of p , $\nu_p(\cdot)$ is an equivalent norm on ℓ_1 .

It is clear that $\nu_p(\cdot)$ is equivalent to the ℓ_1 norm whenever p_n converges to 1 sufficiently quickly. Moreover, it is easy to determine for which sequences p the norm $\nu_p(\cdot)$ is equivalent to the ℓ_1 norm. The characterization is in terms of the dual norm $\nu_q(\cdot)$, where $q = (q_1, q_2, \dots)$ satisfies $\frac{1}{p_n} + \frac{1}{q_n} = 1$ for each $n \in \mathbb{N}$.

Proposition 1. *Let p be a sequence in $(1, \infty)$ and let q be the sequence of conjugate exponents of p . Then the following are equivalent.*

- (i) $\nu_p(\cdot)$ is equivalent to the ℓ_1 norm.
- (ii) $\nu_q(\cdot)$ is equivalent to the c_0 norm.
- (iii) $\lim_{n \rightarrow \infty} \nu_q(1_{[1, n]}) < \infty$.
- (iv) There exists $\delta > 0$ so that for all n , $q_n^\# \geq \delta \log n$, where $(q_n^\#)$ is the increasing rearrangement of q .

Proof. We will use the notation 1_E to denote the characteristic function of a subset E of $\mathbb{N}$. The equivalence of the first three conditions is well-known in a general context. That implication (iii) implies (iv) is straightforward. Indeed, since for each n , there are at least n values of k for which $q_k \leq q_n^\#$, we have for sufficiently large N that

$$\nu_q(1_{[1,N]}) \geq n^{1/q_n^\#}.$$

So, if we set $C := \lim_{N \rightarrow \infty} \nu_q(1_{[1,N]})$, then for all n ,

$$q_n^\# \geq \frac{\log n}{\log C}.$$

For (iv) implies (iii), let $C > 3^{1/\delta}$. Then

$$\sum_{n=1}^{\infty} C^{-q_n} = \sum_{n=1}^{\infty} C^{-q_n^\#} < \infty.$$

Now for $x > 0$, $s > 1$, $(1 + x^{-1})^{1/s} \leq 1 + s^{-1}x^{-1}$, and therefore $(1 + x^s)^{1/s} \leq x + s^{-1}x^{1-s}$. Hence, for each $k \leq N$

$$\nu_q(1_{[k,N]}) \leq \nu_q(1_{[k+1,N]}) + q_k^{-1} \nu_q(1_{[k+1,N]})^{1-q_k}.$$

If $\nu_q(1_{[1,N]}) \leq C$ for all $N \in \mathbb{N}$, then we are done. If $\nu_q(1_{[1,N]}) > C$, choose m so that $\nu_q(1_{[m,N]}) \geq C > \nu_q(1_{[m+1,N]})$. Then

$$\nu_q(1_{[1,N]}) \leq C + 1 + \sum_{k=1}^{m-1} q_k^{-1} \nu_q(1_{[k+1,N]})^{1-q_k} \leq C + 1 + C \sum_{k=1}^{\infty} q_k^{-1} C^{-q_k} < \infty.$$

□

The next result shows the optimality of James's theorem by proving that the above renormings of ℓ_1 fail to contain any asymptotically isometric copies of ℓ_1 .

Theorem 1. *Let $p = (p_n)$ be a sequence in $(1, \infty)$, converging to 1 and such that ν_p is an equivalent norm on ℓ_1 ; and let (ϵ_n) be a null sequence in $(0, 1)$. Then there does not exist a ν_p -normalized sequence (x_k) in ℓ_1 such that, for all $t = (t_j) \in \ell_1$,*

$$\sum_{j=1}^{\infty} (1 - \epsilon_j) |t_j| \leq \nu_p \left(\sum_{j=1}^{\infty} t_j x_j \right) \leq \sum_{j=1}^{\infty} |t_j|.$$

Proof. Without loss of generality, assume p strictly decreases to 1. In order to obtain a contradiction, assume that there exists a null sequence (ϵ_n) in $(0, 1)$ and a ν_p -normalized sequence (x_k) in ℓ_1 such that

$$(*) \quad \sum_{j=1}^{\infty} (1 - \epsilon_j) |t_j| \leq \nu_p \left(\sum_{j=1}^{\infty} t_j x_j \right) \leq \sum_{j=1}^{\infty} |t_j| \quad \text{for all } t = (t_j) \in \ell_1.$$

By passing to a subsequence of (x_n) if necessary, there is no loss of generality in assuming that

$$(**) \quad \sum_{n=1}^{\infty} \epsilon_n < 1.$$

Note also that there is no loss of generality in assuming additionally that the sequence (x_n) is disjointly supported, i.e., that the support of x_m is disjoint from the support of x_n if $m \neq n$. Indeed this is a classical gliding hump argument. Since the closed unit ball of ℓ_1 is weak-star sequentially compact with respect to the predual c_0 , by passing to a subsequence, we may suppose that (x_n) converges weak-star (and so pointwise with respect to the usual basis (e_n) of ℓ_1) to some $y \in \ell_1$. By replacing (x_n) by the ν_p -normalization of the sequence $\left(\frac{x_{2j} - x_{2j+1}}{2}\right)$, we may assume that $y = 0$. As in the proof of the Bessaga-Pełczyński theorem [BP] (or see, for example [D]), by passing to a subsequence of (x_n) which is essentially disjointly supported, truncating to obtain a disjointly supported sequence, and then normalizing, yield a block basis (b_k) of (e_n) which satisfies (*). Consequently, we henceforth assume that (x_n) is disjointly supported.

Let $(m(k))_{k=0}^\infty$ be a strictly increasing sequence in $\mathbb{N} \cup \{0\}$ with $m(0) = 0$ and $(\xi_j)_{j=1}^\infty$ a sequence of scalars such that, for each $k \in \mathbb{N}$,

$$x_k = \sum_{j=m(k-1)+1}^{m(k)} \xi_j e_j .$$

Let N be in $\mathbb{N}$ and, in (*), set $t_j = 1$ for $j = 1, \dots, N$ and 0 otherwise. Then, for $N \geq m(1)$:

$$\begin{aligned} N - \sum_{j=1}^N \epsilon_j &\leq \nu_p \left(\sum_{k=1}^N x_k \right) \\ &= \left(|\xi_1|^{p_1} + \nu_p \left(\sum_{j=2}^{m(N)} \xi_j e_j \right)^{p_1} \right)^{1/p_1} \\ &\leq \left(|\xi_1|^{p_2} + \nu_p \left(\sum_{j=2}^{m(N)} \xi_j e_j \right)^{p_2} \right)^{1/p_2} \\ &= \left(|\xi_1|^{p_2} + |\xi_2|^{p_2} + \nu_p \left(\sum_{j=3}^{m(N)} \xi_j e_j \right)^{p_2} \right)^{1/p_2} \\ &\leq \left(|\xi_1|^{p_3} + |\xi_2|^{p_3} + \nu_p \left(\sum_{j=3}^{m(N)} \xi_j e_j \right)^{p_3} \right)^{1/p_3} \\ &\vdots \\ &\leq \left(|\xi_1|^{p_{m(1)}} + \dots + |\xi_{m(1)}|^{p_{m(1)}} + \nu_p \left(\sum_{j=m(1)+1}^{m(N)} \xi_j e_j \right)^{p_{m(1)}} \right)^{1/p_{m(1)}} \\ &= \left(\|x_1\|_{p_{m(1)}}^{p_{m(1)}} + \nu_p \left(\sum_{k=2}^N x_k \right)^{p_{m(1)}} \right)^{1/p_{m(1)}} \\ &\leq \left(\|x_1\|_{p_{m(1)}}^{p_{m(1)}} + (N-1)^{p_{m(1)}} \right)^{1/p_{m(1)}} . \end{aligned}$$

Thus, for $N \geq m(1)$,

$$\left(N - \sum_{j=1}^N \epsilon_j \right)^{p_{m(1)}} - (N-1)^{p_{m(1)}} \leq \|x_1\|_{p_{m(1)}}^{p_{m(1)}}.$$

By (**), the left-hand side of the inequality tends to ∞ with N . This yields a contradiction which finishes the proof. $\square$

In the previous proof, choosing vectors of the form $x_1 + Mx_N$, instead of $x_1 + \dots + x_N$, also leads to a contradiction (by letting N and then M become arbitrarily large).

We note here that the proof of James's distortion theorem for c_0 gives us that if a Banach space $(X, \|\cdot\|)$ contains an isomorphic copy of c_0 , then for each sequence (ϵ_n) of positive numbers decreasing to 0, there exists a sequence (x_n) in the unit sphere of X such that for each n , $(1 - \epsilon_n) \max_{k \geq n} |t_k| \leq \|\sum_{k=n}^{\infty} t_k x_k\| \leq (1 + \epsilon_n) \max_{k \geq n} |t_k|$, for all $(t_k) \in c_0$. In order to show that James's distortion theorem for c_0 is also optimal, the construction introduced for ℓ_1 can be used as long as the sequence $p = (p_j)$ is chosen to increase sufficiently quickly to infinity. For example, with $p_j = 2^j$,

$$\|x\|_{\infty} \leq \nu_p(x) \leq 2 \|x\|_{\infty} \quad \text{for all } x \in c_0.$$

Thus, with this choice of p , $\nu_p(\cdot)$ is an equivalent norm on c_0 . (For other choices of p , we may apply Proposition 1.)

A Banach space is said to contain an *asymptotically isometric copy* of c_0 if, for every sequence of positive numbers (ϵ_n) decreasing to 0, there exists a sequence (x_n) in the Banach space such that $\max_{n \in F} (1 - \epsilon_n) |\alpha_n| \leq \|\sum_{n \in F} \alpha_n x_n\| \leq \max_{n \in F} (1 + \epsilon_n) |\alpha_n|$ for all choices of scalars (α_n) and for all finite subsets F of natural numbers. (Note that $(1 + \epsilon_n)$ may be replaced by 1 in this definition.) The next result provides a useful connection between the two asymptotically isometric properties.

Theorem 2. *Let $(X, \|\cdot\|)$ be a Banach space that contains an asymptotically isometric copy of c_0 . Then X^* , with the dual norm, contains an asymptotically isometric copy of ℓ_1 .*

Proof. By hypothesis, given any null sequence (ϵ_n) in $(0, 1)$, there is a sequence (x_n) in X such that for all finite sequences of scalars $(\alpha_n)_{n=1}^N$,

$$\max_{1 \leq n \leq N} (1 - \epsilon_n) |\alpha_n| \leq \left\| \sum_{n=1}^N \alpha_n x_n \right\| \leq \max_{1 \leq n \leq N} |\alpha_n|.$$

Let (x_n^*) be a sequence of Hahn-Banach extensions to elements of X^* of the linear functionals on the span of (x_n) that are biorthogonal to (x_n) . Consider x_m^* , for some $m \in \mathbb{N}$. Then, for all vectors x of the form $\sum_{n=1}^N \alpha_n x_n$ with $N \geq m$, we have

$$\begin{aligned} |x_m^*(x)| &= |\alpha_m| = (1 - \epsilon_m)^{-1} (1 - \epsilon_m) |\alpha_m| \\ &\leq (1 - \epsilon_m)^{-1} \max_{1 \leq n \leq N} (1 - \epsilon_n) |\alpha_n| \leq (1 - \epsilon_m)^{-1} \|x\|; \end{aligned}$$

and hence it follows that $\|x_m^*\| \leq (1 - \epsilon_m)^{-1}$.

Set $x'_n := \|x_n^*\|^{-1} x_n^*$ for each $n \in \mathbb{N}$. Fix a sequence $(\alpha_n)_{n=1}^N$ of scalars and let $\beta_n = \text{sign } \alpha_n$ for all n . Then, since $\|\sum_{n=1}^N \beta_n x_n\| \leq \max_{1 \leq n \leq N} |\beta_n| = 1$, we have

that

$$\left\| \sum_{n=1}^N \alpha_n x'_n \right\| \geq \left\langle \sum_{n=1}^N \alpha_n x'_n, \sum_{k=1}^N \beta_k x_k \right\rangle = \sum_{n=1}^N \|x_n^*\|^{-1} |\alpha_n| \geq \sum_{n=1}^N (1 - \epsilon_n) |\alpha_n| .$$

Thus, X^* contains an asymptotically isometric copy of ℓ_1 . $\square$

Combining Theorems 1 and 2, we immediately get that not every renorming of c_0 contains an asymptotically isometric copy of c_0 .

Theorem 3. *Let $q = (q_n)$ be a sequence in $(1, \infty)$, diverging to ∞ and such that ν_q is an equivalent norm on c_0 ; and let (ϵ_n) be a null sequence in $(0, 1)$. Then there does not exist a sequence (x_k) in c_0 such that, for all $\alpha = (\alpha_j) \in c_0$,*

$$\max_{n \in F} (1 - \epsilon_n) |\alpha_n| \leq \nu_q \left(\sum_{n \in F} \alpha_n x_n \right) \leq \max_{n \in F} (1 + \epsilon_n) |\alpha_n|$$

for all finite subsets F of natural numbers.

Proof. It is enough to apply Theorems 1 and 2, after noting that the dual of (c_0, ν_q) is (ℓ_1, ν_p) , where p is the sequence of conjugate exponents of q . $\square$

In closing, note that other renormings of ℓ_1 exist that fail to contain asymptotically isometric copies of ℓ_1 . One such norm is:

$$\|x\|' := \sup_{n \in \mathbb{N}} \gamma_n \sum_{k=n}^{\infty} |\xi_k| , \text{ for all } x = (\xi_n) \in \ell_1 ,$$

where (γ_n) is a fixed sequence in $(0, 1)$ that strictly increases to 1. The details needed to show that ℓ_1 with this norm fails to contain asymptotically isometric copies of ℓ_1 are similar to those given for the ν_p -norm. (Although, when checking the analogue of the proof of Theorem 1 for the $\|\cdot\|'$ -norm, (after assuming, without loss of generality, that (ϵ_n) decreased to 0 sufficiently fast), we used the sequence $(x_1 + Nx_N)_N$ instead of $(x_1 + \cdots + x_N)_N$.)

Whether ℓ_1 endowed with either of the norms $\|\cdot\|'$ or ν_p has the fixed point property is unknown. The norm $\|\cdot\|'$, suggested to us by the referee of another paper, is interesting because of its link to the strengthening of James's distortion theorem described earlier.

Finally, let us consider c_0 . It is shown in [DLT] that whenever a Banach space $(X, \|\cdot\|)$ contains an asymptotically isometric copy of c_0 , it must fail the FPP. We remark that the spaces (c_0, ν_q) of Theorem 3 *also* fail the FPP. Indeed, without loss of generality, assume (q_n) increases to ∞ . Then a fixed point free ν_q -nonexpansive map T on a closed, bounded and convex set C is provided by the usual c_0 example: i.e. let $C := \{x = (\xi_n) \in c_0 : 0 \leq \xi_n \leq 1 \text{ for all } n\}$ and define $T(x) := (1, \xi_1, \xi_2, \xi_3, \dots)$.

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