DENSITIES WITH THE MEAN VALUE PROPERTY FOR HARMONIC FUNCTIONS IN A LIPSCHITZ DOMAIN

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ABSTRACT. Let D be a bounded domain in $\mathbb{R}^n$, $n \geq 2$, and let $x_0 \in D$. We consider positive functions w on D such that $h(x_0) = (\int_D w dx)^{-1} \int_D h w dx$ for all bounded harmonic functions h on D . We determine Lipschitz domains D having such w with $\inf_D w > 0$.

1. INTRODUCTION

Let D be a bounded domain in $\mathbb{R}^n$, $n \geq 2$, and let $x_0 \in D$. By $\mathcal{H}_b$ we denote the set of all bounded harmonic functions on D . Hansen and Netuka [3] considered positive functions w on D such that

$$(1) \quad h(x_0) = \frac{1}{\int_D w dx} \int_D h w dx \quad \text{for } h \in \mathcal{H}_b.$$

They proved the following theorem.

- Theorem A.** (i) *There exists a domain D which fails to have w satisfying (1) and $\inf_D w > 0$.*
 (ii) *For every bounded domain D there is positive $w \in C^\infty(D)$ satisfying (1) and $\lim_{x \rightarrow \partial D} w(x) = 0$.*
 (iii) *For a $C^{1+\varepsilon}$ -domain (or a Liapunov-Dini domain (cf. [5])) there is $w \in C^\infty(D)$ satisfying (1) and $\inf_D w > 0$.*

We say that a bounded domain D is k -Lipschitz if D and ∂D are given locally by a Lipschitz function whose Lipschitz constant is at most k . If D is k -Lipschitz for some $k > 0$, then we say that D is a Lipschitz domain. In this note we consider the same problem for a Lipschitz domain. Our result is the following.

- Theorem.** (i) *There exists a $1/\sqrt{n-1}$ -Lipschitz domain which fails to have w satisfying (1) and $\inf_D w > 0$.*
 (ii) *If $0 < k < 1/\sqrt{n-1}$, then for every k -Lipschitz domain D there is $w \in C^\infty(D)$ satisfying (1) and $\inf_D w > 0$.*

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lemma which A. Ancona communicated to him. The author would like to express his sincere gratitude to Professor Hansen.

2. PROOF OF THEOREM (i)

Let $T_\psi = \{x = (x_1, \dots, x_n) : x_n > |x| \cos \psi\}$. This is an infinite cone with vertex at 0 and aperture ψ . By u_ψ (resp. v_ψ) we denote the Martin kernel for T_ψ with pole at ∞ (resp. at 0). It is known that u_ψ (resp. v_ψ) is of homogeneous of degree $\alpha = \alpha_n(\psi)$ (resp. of degree $2 - n - \alpha$). The constant $\alpha_n(\psi)$ is called the maximal order of barriers. It is not so difficult to see that α_n is strictly decreasing; $\alpha_n(\pi/2) = 1$; $\alpha_n(\psi) = 2$ for $\cos \psi = 1/\sqrt{n}$; $\lim_{\psi \rightarrow 0} \alpha_n(\psi) = \infty$; $\lim_{\psi \rightarrow \pi} \alpha_n(\psi) = 0$ (for $n \geq 3$); $\alpha_2(\psi) = \pi/(2\psi)$; and $\alpha_4(\psi) = \pi/\psi - 1$. For the above result see [1] and the references therein. By $B(x, r)$ we denote the open ball with center at x and radius r .

Proof of Theorem (i). In this proof we let $\cos \psi = 1/\sqrt{n}$. Then $\tan \psi = \sqrt{n-1}$ and

$$T_\psi = \{x = (x', x_n) : x_n > \frac{1}{\sqrt{n-1}}|x'|\}.$$

Hence it is easy to construct a $1/\sqrt{n-1}$ -Lipschitz domain U_ψ such that $U_\psi \cap B(0, 1) = T_\psi \cap B(0, 1)$. Let $x_0 \in U_\psi$ be a fixed point and let K_0 be the Martin kernel for U_ψ with pole at 0 with reference point at x_0 . We observe that K_0 is comparable with v_ψ near the origin. Since $\alpha_n(\psi) = 2$, it follows that v_ψ is of homogeneous of degree $-n$ and hence K_0 is not integrable over U_ψ . Put

$$K_j = \frac{\omega(\cdot, B(0, 1/j) \cap \partial U_\psi)}{\omega(x_0, B(0, 1/j) \cap \partial U_\psi)},$$

where ω denotes the harmonic measure for U_ψ . Then $K_j \in \mathcal{H}_b(U_\psi)$, $K_j(x_0) = 1$ and $K_j \rightarrow K_0$ locally uniformly on U_ψ . By Fatou's lemma

$$\liminf_{j \rightarrow \infty} \int_{U_\psi} K_j dx \geq \int_{U_\psi} K_0 dx = \infty.$$

Hence

$$\sup \left\{ \int_{U_\psi} h dx : h \in \mathcal{H}_b(U_\psi), h(x_0) = 1 \right\} = \infty.$$

This implies that U_ψ fails to have w satisfying (1) and $\inf_{U_\psi} w > 0$ (cf. [3, Corollary 2.3]). The theorem is proved. $\square$

3. ESTIMATE OF THE GRADIENT OF HARMONIC FUNCTIONS

By the symbol c we denote a positive constant whose value is unimportant and may change from line to line. If necessary, we use $c_1, c_2, \dots$, to specify them. We shall say that two positive functions f_1 and f_2 are comparable, written $f_1 \approx f_2$, if and only if there exists a constant c such that $c^{-1}f_2 \leq f_1 \leq cf_2$.

In this section we shall give a certain lower estimate of the gradient of a positive harmonic function. This estimate will be used for the proof of Theorem (ii). The estimate is based on the following general lemma of A. Ancona, about which Professor Hansen kindly informed the author. The present form of the lemma and its proof are due to Professor Hansen.

Lemma 1. *Let $0 < c_1 < 1$, $0 < c_2 < 1$ and $0 < A < 1$. Then there exists a , $0 < a < 1$, depending only on c_1 , c_2 and A such that for any positive harmonic function u on $B(0, 1)$ with $u(0) = 1$ and $\inf u(B(0, c_2)) \leq 1 - A$, we have $\inf u(B(0, c_1)) \leq 1 - a$.*

Proof. It suffices to consider the case $0 < c_1 < c_2 < 1$. Suppose the contrary. Then there exists a sequence $\{u_j\}$ of positive harmonic functions on $B(0, 1)$ such that $u_j(0) = 1$, $\inf u_j(B(0, c_2)) \leq 1 - A$ and $\lim_{j \rightarrow \infty} (\inf u_j(B(0, c_1))) = 1$. Taking a locally convergent subsequence, we obtain a positive harmonic function u on $B(0, 1)$ such that $\inf u(B(0, c_2)) \leq 1 - A$ and $u(0) = 1 = \inf u(B(0, c_1))$. By the minimum principle $u = 1$ on $B(0, c_1)$ and hence on $B(0, 1)$ by analyticity. Thus a contradiction arises. $\square$

By translation and dilation we obtain the following corollary.

Corollary. *Let $0 < c_1 < 1$, $0 < c_2 < 1$ and $0 < A < 1$ and let $0 < a < 1$ be as in Lemma 1. Let $x \in \mathbb{R}^n$ and $r > 0$. Then for any positive harmonic function u on $B(x, r)$ with $\inf u(B(x, c_2 r)) \leq (1 - A)u(x)$, we have $\inf u(B(x, c_1 r)) \leq (1 - a)u(x)$.*

From this corollary we obtain the following lower estimate of the gradient of a positive harmonic function, which may be of some independent interest. By $m(E)$ we denote the Lebesgue measure of E .

Lemma 2 (cf. [1, Lemma 1]). *Let $0 < c_1 < 1$, $0 < c_2 < 1$ and $0 < A < 1$. Then there exist positive constants ε and c_3 depending only on c_1 , c_2 and A with the following property: Let $x \in \mathbb{R}^n$ and $r > 0$. If u is a positive harmonic function on $B(x, r)$ and there is $x^* \in B(x, c_2 r)$ such that $u(x^*) \leq (1 - A)u(x)$, then*

$$m(\{y \in B(x, c_1 r) : |\nabla u(y)| \geq \varepsilon \frac{u(x)}{r}\}) \geq c_3 r^n.$$

Moreover, for $1 \leq p < \infty$, there exists a positive constant c_p depending only on c_1 , c_2 , A and p such that

$$u(x)^p \leq c_p r^{p-n} \int_{B(x, c_1 r)} |\nabla u|^p dy.$$

Proof. Let a be as in Lemma 1. Then, by Corollary, we find $x' \in \overline{B}(x, c_1 r)$ such that $u(x') \leq (1 - a)u(x)$. By the Harnack inequality, we find c_4 , $0 < c_4 < c_1$, depending only on a such that $u(y) \geq (1 - a/2)u(x)$ for $y \in B(x, c_4 r)$. Hence

$$(2) \quad u(y) - u(x') \geq \frac{a}{2}u(x) \quad \text{for } y \in B(x, c_4 r).$$

Let ℓ_y be the line segment connecting y and x' for $y \in B(x, c_4 r)$. We observe that $\ell_y \subset \overline{B}(x, c_1 r)$ and so the length $|\ell_y|$ of ℓ_y is uniformly bounded by $2c_1 r$. On the other hand, representing u as the Poisson integral and differentiating under the integral sign, we obtain

$$(3) \quad |\nabla u(z)| \leq c_5 \frac{u(x)}{r} \quad \text{for } z \in B(x, c_1 r).$$

Now let $\varepsilon = a/(8c_1)$. We have from (2) and (3)

$$\begin{aligned} \frac{a}{2}u(x) &\leq u(y) - u(x') \leq \int_{\ell_y} |\nabla u| ds \\ &\leq \int_{\{z \in \ell_y : |\nabla u(z)| < \varepsilon u(x)/r\}} \varepsilon \frac{u(x)}{r} ds + \int_{\{z \in \ell_y : |\nabla u(z)| \geq \varepsilon u(x)/r\}} c_5 \frac{u(x)}{r} ds \\ &\leq 2c_1 r \varepsilon \frac{u(x)}{r} + c_5 \frac{u(x)}{r} |\{z \in \ell_y : |\nabla u(z)| \geq \varepsilon \frac{u(x)}{r}\}| \\ &= \frac{a}{4}u(x) + c_5 \frac{u(x)}{r} |\{z \in \ell_y : |\nabla u(z)| \geq \varepsilon \frac{u(x)}{r}\}| \end{aligned}$$

uniformly for $y \in B(x, c_4 r)$. Hence

$$|\{z \in \ell_y : |\nabla u(z)| \geq \varepsilon \frac{u(x)}{r}\}| \geq \frac{a}{4c_5} r.$$

Since the above inequality holds uniformly for $y \in B(x, c_4 r)$, Fubini's theorem yields the required estimate.

Now we integrate $|\nabla u|$ over $B(x, c_1 r)$. We have

$$\int_{B(x, c_1 r)} |\nabla u| dy \geq \varepsilon \frac{u(x)}{r} c_3 r^n,$$

whence, by Hölder's inequality, for $1 \leq p < \infty$,

$$\frac{u(x)}{r} \leq c \frac{1}{m(B(x, c_1 r))} \int_{B(x, c_1 r)} |\nabla u| dy \leq c \left(\frac{1}{m(B(x, c_1 r))} \int_{B(x, c_1 r)} |\nabla u|^p dy \right)^{1/p}.$$

This readily yields the second assertion. $\square$

4. PROOF OF THEOREM (ii)

Hereafter, we let D be a k -Lipschitz domain with $0 < k < 1/\sqrt{n-1}$. By $\delta(x)$ we denote the distance $\text{dist}(x, \mathbb{C}D)$. Let us recall that $x_0 \in D$ is a fixed point and put $g(x) = G(x, x_0)$, where G is the Green function for D . The function g is positive and harmonic in $D \setminus \{x_0\}$ and vanishes on ∂D . Let $r_0 = \frac{1}{3}\delta(x_0)$. It is easy to see that $x_0 \notin B(\xi, 3r_0)$ for any $\xi \in \partial D$.

Lemma 3. *There exists a constant c_6 , $0 < c_6 < 1$, depending only on D such that if $x \in D$ and $\delta(x) < r_0$, then there is $x^* \in B(x, c_6 \delta(x))$ such that $g(x^*) \leq \frac{1}{2}g(x)$.*

Proof. Let $x \in D$ and $\delta(x) < r_0$. Take $\bar{x} \in \partial D$ such that $|x - \bar{x}| = \delta(x)$. We observe that g is a positive harmonic function on $D \cap B(\bar{x}, 3\delta(x))$ which vanishes on ∂D . The boundary Harnack principle (e.g. [4, Lemma 4.10]) yields

$$g \approx g(x)\omega(\cdot, D \cap \partial B(\bar{x}, 2\delta(x))) \quad \text{on } D \cap B(\bar{x}, \delta(x)),$$

where ω denotes the harmonic measure. A Lipschitz domain is uniformly Δ -regular (cf. [2]) and there is $\kappa > 0$ depending only on D such that for $0 < \rho < 2$

$$\omega(\cdot, D \cap \partial B(\bar{x}, 2\delta(x))) \leq c\rho^\kappa \quad \text{on } D \cap B(\bar{x}, \rho\delta(x)).$$

Hence, if ρ is sufficiently small, then

$$g \leq \frac{1}{2}g(x) \quad \text{on } D \cap B(\bar{x}, \rho\delta(x));$$

in particular, we find $x^* \in B(x, (1 - \rho/2)\delta(x))$ such that $g(x^*) \leq \frac{1}{2}g(x)$. Thus the lemma follows with $c_6 = 1 - \rho/2$. $\square$

For $x \in D$ we let

$$B^*(x) = \{y \in D : |x - y| < \frac{1}{2}\delta(y)\}.$$

Then, it is easy to see that

$$B(x, \frac{1}{3}\delta(x)) \subset B^*(x).$$

In fact, if $y \in B(x, \frac{1}{3}\delta(x))$, then $|x - y| < \frac{1}{3}\delta(x)$ and $\frac{2}{3}\delta(x) < \delta(y) < \frac{4}{3}\delta(x)$, whence $|x - y| < \frac{1}{2}\delta(y)$. With the aid of Lemma 3, we can invoke Lemma 2 with $c_1 = \frac{1}{3}$, $c_2 = c_6$, $A = \frac{1}{2}$, $r = \delta(x)$, $u = g$ and $p = 2$. We have

$$(4) \quad \int_{B^*(x)} |\nabla g|^2 dy \geq \int_{B(x, \frac{1}{3}\delta(x))} |\nabla g|^2 dy \geq c\delta(x)^{n-2}g(x)^2$$

for $x \in D$ with $\delta(x) < r_0$. Now we are ready to prove Theorem (ii).

Proof of Theorem (ii). We find $\gamma > 0$ such that

$$\{x \in D : g(x) < \gamma\} \subset \{x \in D : \delta(x) < r_0\}.$$

As was observed in Section 2, the function u_ψ is homogeneous of degree 2 for $\cos \psi = 1/\sqrt{n}$. By the comparison of g and u_ψ , we can find α , $1 < \alpha < 2$, such that

$$(5) \quad g(x) \geq c\delta(x)^\alpha \quad \text{for } x \in D,$$

since the domain D is k -Lipschitz with $0 < k < 1/\sqrt{n-1}$. Let $\eta = (2 - \alpha)/\alpha$ and define

$$w^*(x) = \begin{cases} |\nabla g(x)|^2 g(x)^{-2} & \text{if } g(x) \geq \gamma, \\ |\nabla g(x)|^2 g(x)^{\eta-1} & \text{if } g(x) < \gamma. \end{cases}$$

Let $h \in \mathcal{H}_b$. In the same way as in [1, Section 2], we have from the coarea formula and the Poisson integral formula

$$\begin{aligned} \int_D h w^* dx &= \int_0^\gamma dt \int_{\{g=t\}} h |\nabla g|^2 g^{\eta-1} \frac{d\sigma}{|\nabla g|} + \int_\gamma^\infty dt \int_{\{g=t\}} h |\nabla g|^2 g^{-2} \frac{d\sigma}{|\nabla g|} \\ (6) \quad &= \int_0^\gamma t^{\eta-1} dt \int_{\{g=t\}} h |\nabla g| d\sigma + \int_\gamma^\infty t^{-2} dt \int_{\{g=t\}} h |\nabla g| d\sigma \\ &= h(x_0) \left(\int_0^\gamma t^{\eta-1} dt + \int_\gamma^\infty t^{-2} dt \right) = \left(\frac{\gamma^\eta}{\eta} + \frac{1}{\gamma} \right) h(x_0). \end{aligned}$$

Next we mollify w^* . Fix a radially symmetric smooth function $\tau \geq 0$ on $\mathbb{R}^n$ such that $\{\tau > 0\} = B(0, 1)$ and $\int_{\mathbb{R}^n} \tau dx = 1$. For $y \in D$ we define

$$\tau_y : x \mapsto \frac{1}{(\frac{2}{3}\delta(y))^n} \tau\left(\frac{x - y}{\frac{2}{3}\delta(y)}\right), \quad x \in \mathbb{R}^n.$$

Let

$$w(x) = \int_D \tau_y(x) w^*(y) dy.$$

It is easy to see that w is a positive smooth function on D . By Fubini's theorem, (6) and the mean value property of $h \in \mathcal{H}_b$, we have

$$\begin{aligned} \int_D h(x)w(x)dx &= \int_D h(x)dx \int_D \tau_y(x)w^*(y)dy \\ &= \int_D w^*(y)dy \int_D h(x)\tau_y(x)dx \\ &= \int_D w^*(y)h(y)dy = \left(\frac{\gamma^\eta}{\eta} + \frac{1}{\gamma}\right)h(x_0). \end{aligned}$$

Thus w satisfies (1). Let

$$V = \{x \in D : g < \gamma \text{ on } B^*(x)\}.$$

We observe that $D \setminus V$ is a compact subset of D and $w \geq c$ on $D \setminus V$. Hence we have only to show

$$w \geq c \quad \text{on } V.$$

We observe that

$$\tau_y(x) = \frac{1}{(\frac{2}{3}\delta(y))^n} \tau\left(\frac{x-y}{\frac{2}{3}\delta(y)}\right) \geq \frac{\inf \tau(B(0, \frac{3}{4}))}{(\frac{2}{3}\delta(y))^n} \geq c\delta(x)^{-n}$$

for $y \in B^*(x) = \{y : |x-y| < \frac{1}{2}\delta(y)\}$. Suppose $x \in V$. Then $w^* = |\nabla g|^2 g^{\eta-1} \approx |\nabla g|^2 g(x)^{\eta-1}$ on $B^*(x)$ by the Harnack inequality. It follows from (4) that

$$\begin{aligned} w(x) &\geq \int_{B^*(x)} \tau_y(x)w^*(y)dy \geq c\delta(x)^{-n}g(x)^{\eta-1} \int_{B^*(x)} |\nabla g|^2 dy \\ &\geq c\delta(x)^{-n}g(x)^{\eta-1}\delta(x)^{n-2}g(x)^2 = cg(x)^{\eta+1}\delta(x)^{-2}. \end{aligned}$$

By (5) and the definition of η we have

$$w(x) \geq cg(x)^{\eta+1}\delta(x)^{-2} \geq c\delta(x)^{\alpha(\eta+1)-2} = c.$$

The theorem follows. □

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