A GENERALIZATION OF THE CLASSICAL SPHERE THEOREM

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ABSTRACT. In this paper, we prove a sphere theorem for Riemannian manifolds with partially positive curvature which generalizes the classical sphere theorem.

1. INTRODUCTION

The celebrated Rauch-Berger-Klingenberg sphere theorem [1] states that, if M is an n -dimensional complete simply connected Riemannian manifold with sectional curvature K_M satisfying $1 < K_M \leq 4$, then M is homeomorphic to an n -sphere S^n . In 1977, Grove and Shiohama [2] generalized this sphere theorem to the following form: If the sectional curvature and diameter of M satisfy $d(M) > \frac{\pi}{2}$ and $K_M \geq 1$ then M is homeomorphic to S^n . In this paper, we shall prove a sphere theorem for manifolds with partially positive curvature which also generalizes the Rauch-Berger-Klingenberg sphere theorem.

Before stating our results, we fix some notations for the k -th Ricci curvature of a Riemannian manifold (Cf. [4], [6]). Let M be an n -dimensional Riemannian manifold. If, for any point $x \in M$ and any $(k+1)$ -mutually orthogonal unit tangent vectors $e, e_1, \dots, e_k \in T_x M$, we have $\sum_{i=1}^k K(e \wedge e_i) \geq kc$ (resp., $> kc$), we say that the k -th Ricci curvature of M is not less than kc (resp., larger than kc) and denote this fact by $\text{Ric}_{(k)}(M) \geq kc$ (resp., $> kc$). Here, $K(e \wedge e_i)$ denotes the sectional curvature of the plane spanned by e and e_i ($1 \leq i \leq k$). Thus, $\text{Ric}_{(1)}(M) \geq c$ (resp., $> c$) is equivalent to $K_M \geq c$ (resp., $> c$) and $\text{Ric}_{(n-1)}(M) \geq (n-1)c$ (resp., $> (n-1)c$) is equivalent to the fact that the Ricci curvature of M satisfies $\text{Ric}_M \geq (n-1)c$ (resp., $> (n-1)c$). Also, it is easy to see that $\text{Ric}_{(k)}(M) \geq kc$ (resp., $> kc$) implies $\text{Ric}_{(m)}(M) \geq mc$ (resp., $> mc$) for $m \geq k$.

Now we can state our main theorem as follows.

Theorem 1. *Let M be an $n(\geq 3)$ -dimensional complete simply connected Riemannian manifold with $\text{Ric}_{(\lfloor \frac{n+1}{2} \rfloor)}(M) > [\frac{n+1}{2}]$. If there exists a point $p \in M$ such that the injectivity radius i_p of M at p satisfies $i_p \geq \frac{\pi}{2}$, then M is homeomorphic to an n -sphere S^n .*

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Let $M = CP^2(4)$ be the complex projective space of complex dimension 2 and of holomorphic sectional curvature 4. Then the injectivity radius of M is $\frac{\pi}{2}$ and $\text{Ric}_{(3)}(M) = 6 > 3$, but M is not homeomorphic to S^4 . This example shows that the curvature condition “ $\text{Ric}_{(\lfloor \frac{n+1}{2} \rfloor)}(M) > \lfloor \frac{n+1}{2} \rfloor$ ” in Theorem 1 can't be weakened to “ $\text{Ric}_{(1+\lfloor \frac{n+1}{2} \rfloor)}(M) > 1 + \lfloor \frac{n+1}{2} \rfloor$ ”.

We remark that when $n = 3$, Theorem 1 is a special case of Hamilton's theorem [3]. An important result of Hartman [4] states that if M is an $n(\geq 3)$ -dimensional complete simply connected Riemannian manifold with $K_M \leq 4$ and $\text{Ric}_{(n-2)}(M) \geq n - 2$, then the injectivity radius of M satisfies $i(M) \geq \frac{\pi}{2}$. Thus, Theorem 1 immediately implies the following generalized form of the classical sphere theorem:

Theorem 2. *Let M be an $n(\geq 3)$ -dimensional complete simply connected Riemannian manifold. If $K_M \leq 4$ and $\text{Ric}_{(\lfloor \frac{n+1}{2} \rfloor)}(M) > \lfloor \frac{n+1}{2} \rfloor$, then M is homeomorphic to S^n .*

2. A PROOF OF THEOREM 1

Before proving Theorem 1, we list some basic facts we need. Let M be a compact n -dimensional Riemannian manifold with boundary. For any $1 \leq k \leq n - 1$, let $\Lambda_k : \partial M \rightarrow R$ be the function on ∂M defined by $\Lambda_k(p) =$ the minimum of all sums of any k eigenvalues of the second fundamental form of ∂M at p with respect to the outward-pointing unit normal N at p . ∂M is called k -convex if $\Lambda_k > 0$ on ∂M .

Lemma 1 ([6]). *Let M be an n -dimensional compact Riemannian manifold with boundary ∂M and let k be an integer satisfying $1 \leq k \leq n - 1$. If $\text{Ric}_{(k)}(M) > 0$ and M has k -convex boundary, then M has the homotopy type of a CW complex obtained from ∂M by attaching a finite number of cells each of dimension $\geq n - k + 1$.*

Proof of Theorem 1. When $n = 3$, Theorem 1 follows from Hamilton's theorem [3]. Thus, we can assume $n \geq 4$. Let $B(p, r)$ be the open geodesic ball of radius r with center p . Since the injectivity radius of M at p is not less than $\frac{\pi}{2}$, thus, for any $r \in (0, \frac{\pi}{2})$, $B(p, r)$ is homeomorphic to an n -disk and the boundary $\partial B(p, r)$ of $B(p, r)$ is a smooth hypersurface and it is homeomorphic to an $(n - 1)$ -sphere S^{n-1} . From $\text{Ric}_{(\lfloor \frac{n+1}{2} \rfloor)}(M) > \lfloor \frac{n+1}{2} \rfloor$, we know that $\text{Ric}_M > n - 1$, thus M is compact by Bonnet-Myers theorem [1], and so we can find a positive constant δ such that $\text{Ric}_{(\lfloor \frac{n+1}{2} \rfloor)}(M) > \lfloor \frac{n+1}{2} \rfloor + \delta$. Take $l = \frac{\pi}{2\sqrt{1 + \frac{\delta}{\lfloor \frac{n+1}{2} \rfloor}}}$; then $l < \frac{\pi}{2}$. Now, we

want to show that $M - B(p, l)$ has $\lfloor \frac{n+1}{2} \rfloor$ -convex boundary. To do this, we denote by ∇ the Riemannian connection of $M - B(p, l)$ and S the second fundamental form of $M - B(p, l)$ with respect to the outward-pointing unit normal N , i.e., $S(u, v) = \langle \nabla_u N, v \rangle$ for $u, v \in T\partial B(p, l)$. Let q be any point on the boundary of $M - B(p, l)$ and $\gamma : [0, l] \rightarrow \overline{B(p, l)}$ be a normal geodesic from p to q that realizes the minimum distance between them. An argument of the first variation of arc-length shows that γ strikes $\partial B(p, l)$ orthogonally. For any k orthonormal vectors $e_1, \dots, e_k$ in $T_q \partial B(p, l)$, we denote by $E_i(s)$ the parallel translate of e_i to $\gamma(s)$ along γ and define $W_i(s) = \sin \frac{\pi s}{2l} E_i(s)$, $i = 1, \dots, k$. Each vector field W_i gives rise to a variation with length $L_{W_i}(t)$ ($t \in (-\epsilon, \epsilon)$) of the variational curves of the geodesic γ keeping one end point p fixed and other end points on $\partial B(p, l)$. The first variation of the arc-length $L'_{W_i}(0) = 0$, and the second variation of the arc-length $L''_{W_i}(0) \geq 0$, $i = 1, \dots, k$, since γ is a minimizing geodesic from p to q . It

then follows from the second variation formula of arc-length (see p.99 in [5]) that for $i = 1, \dots, k$,

$$\begin{aligned}
 (1) \quad 0 &\leq L''_{W_i}(0) \\
 &= \langle \nabla_{e_i} e_i, \gamma'(l) \rangle + \int_0^l \{ |\nabla_{\gamma'(s)} W_i(s)|^2 \\
 &\quad - \langle R(\gamma'(s), W_i(s)) W_i(s), \gamma'(s) \rangle \} ds \\
 &= S(e_i, e_i) + \int_0^l \{ |\nabla_{\gamma'(s)} W_i(s)|^2 \\
 &\quad - \langle R(\gamma'(s), W_i(s)) W_i(s), \gamma'(s) \rangle \} ds,
 \end{aligned}$$

where R is the curvature tensor of M .

Thus, from $\text{Ric}_{(\lfloor \frac{n+1}{2} \rfloor)} > \lfloor \frac{n+1}{2} \rfloor + \delta$ and (1), we get

$$\begin{aligned}
 (2) \quad \sum_{i=1}^{\lfloor \frac{n+1}{2} \rfloor} S(e_i, e_i) &\geq - \sum_{i=1}^{\lfloor \frac{n+1}{2} \rfloor} \int_0^l \{ |\nabla_{\gamma'(s)} W_i(s)|^2 \\
 &\quad - \langle R(\gamma'(s), W_i(s)) W_i(s), \gamma'(s) \rangle \} ds \\
 &= - \sum_{i=1}^{\lfloor \frac{n+1}{2} \rfloor} \int_0^l \{ \frac{\pi^2}{4l^2} \cos^2 \frac{\pi s}{2l} \\
 &\quad - \sin^2 \frac{\pi s}{2l} \langle R(\gamma'(s), E_i(s)) E_i(s), \gamma'(s) \rangle \} ds \\
 &> - \lfloor \frac{n+1}{2} \rfloor \int_0^l (\frac{\pi^2}{4l^2} \cos^2 \frac{\pi s}{2l} - (1 + \frac{\delta}{\lfloor \frac{n+1}{2} \rfloor}) \sin^2 \frac{\pi s}{2l}) ds \\
 &= - \lfloor \frac{n+1}{2} \rfloor \cdot \frac{2l}{\pi} \int_0^{\frac{\pi}{2}} (\frac{\pi^2}{4l^2} \cos^2 s - (1 + \frac{\delta}{\lfloor \frac{n+1}{2} \rfloor}) \sin^2 s) ds \\
 &= - \lfloor \frac{n+1}{2} \rfloor \cdot \frac{l}{2} (\frac{\pi^2}{4l^2} - (1 + \frac{\delta}{\lfloor \frac{n+1}{2} \rfloor})) \\
 &= 0.
 \end{aligned}$$

Therefore, $M - B(p, l)$ has $\lfloor \frac{n+1}{2} \rfloor = n - \lfloor \frac{n}{2} \rfloor$ -convex boundary. From Lemma 1 we conclude that $M - B(p, l)$ has the homotopy type of a CW complex obtained from $\partial B(p, l) \simeq S^{n-1}$ by attaching a finite number of cells each of dimension $\geq \lfloor \frac{n}{2} \rfloor + 1$. So does M since $M = B(p, l) \cup (M - B(p, l))$ and $B(p, l) \simeq D^n$, the n -disk. Thus, $\pi_i(M, B(p, l)) = 0$ for $1 \leq i \leq \lfloor \frac{n}{2} \rfloor$. From the homotopy exact sequence we know that $\pi_i(M) = 0$ for $1 \leq i \leq \lfloor \frac{n}{2} \rfloor$. By a basic argument in topology, one can deduce that M is homeomorphic to the sphere S^n . This completes the proof of Theorem 1. $\square$

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