PROJECTIVE STRUCTURES ON MODULI SPACES OF COMPACT COMPLEX HYPERSURFACES

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ABSTRACT. It is shown that moduli spaces of complete families of compact complex hypersurfaces in complex manifolds often come equipped canonically with projective structures satisfying some natural integrability conditions.

1. PROJECTIVE CONNECTIONS

Let M be a complex manifold. Consider the following equivalence relation on the set of a ne torsion-free connections on M: two connections $\widehat{\ }$ and are said to be projectively equivalent if they have the same geodesics, considered as unparameterized paths. In a local coordinate chart $\{t^\alpha\}$, $\alpha = 1, \ldots, \dim M$ on M, where $\widehat{\ }$ and are represented by Christo el symbols $\widehat{\ }^\gamma_{\alpha\beta}$ and $^\gamma_{\alpha\beta}$ respectively, this equivalence relation reads [H]

$$\widehat{\ } \sim$$

if

$$\widehat{\ }^\gamma_{\alpha\beta} = {}^\gamma_{\alpha\beta} b_\beta \delta^\gamma_\alpha + b_\alpha \delta^\gamma_\beta$$

for some 1-form $b = b_\alpha dt^\alpha$. An equivalence class of torsion-free a ne connections under this relation is called a projective structure or a projective connection.

Let M be a complex manifold with a projective structure. A complex submanifold $P \subset M$ is called *totally geodesic* if for each point $t \subset P$ and each direction tangent to P at t the corresponding geodesic of the projective connection is contained in P at least locally.

2. MODULI SPACES OF COMPACT COMPLEX HYPERSURFACES

Let X be be a compact complex hypersurface in a complex manifold Y with normal line bundle N such that $H^1(X, N) = 0$. According to Kodaira [K-1], such a hypersurface X belongs to the complete analytic family $\{X_t | t \in M\}$ of compact complex hypersurfaces X_t in Y with the moduli space M being a $(\dim_\mathbb{C} H^0(X, N))$-dimensional complex manifold. Moreover, there is a canonical isomorphism $k_t : T_t M \to H^0(X_t, N_t)$ which associates a global section of the normal bundle N_t of $X_t \hookrightarrow Y$ to any tangent vector at the corresponding point $t \in M$.

Consider $F = \{(y,t) \in Y \times M | y \in X_t\}$ and denote by $\mu : F \to Y$ and $\nu : F \to M$ two natural projections,

$$(1) \qquad\qquad\qquad Y \xleftarrow{\mu} F \xrightarrow{\nu} M.$$

The space F is a submanifold in $Y \times M$. If N_F is the normal bundle of $F \hookrightarrow Y \times M$, then, for any point $t \in M$, we have $N_F|_{\nu^{-1}(t)} \simeq N_{X_t|Y}$, where $N_{X_t|Y}$ is the normal bundle of the submanifold $\mu \circ \nu^{-1}(t) = X_t \hookrightarrow Y$. By Kodaira's theorem, there is an isomorphism $k : TM \to \nu_*^0(N_F)$, where $\nu_*^0(N_F)$ denotes the direct image.

Let us denote the point in the moduli space M corresponding to X by t_0, i.e. $X = \mu \circ \nu^{-1}(t_0)$. It is easy to show that, for each $y \in Y' \equiv \bigcup_{t \in M} X_t$ the set $\nu \circ \mu^{-1}(y)$ is a complex analytic subspace of M. We denote by P_y its manifold content, i.e. $P_y = \nu \circ \mu^{-1}(y) \setminus \{\text{singular points}\}$. If the natural evaluation map

$$H^0(X_t, N_{X_t|Y}) \to N_z$$
$$\phi \to \phi(z),$$

where N_z is the bre of N at a point $z \in X_t$ and $\phi(z)$ is the value of the global section $\phi \in H^0(X_t, N_{X_t|Y})$ at z, is surjective at all points $z \in X_t$ and for all $t \in M$, then $P_y = \nu \circ \mu^{-1}(y)$.

3. The main theorem

The idea of studying di erential geometry on the moduli space of compact complex submanifolds of a given ambient complex manifold goes back to Penrose [Pe] who discovered self-dual conformal structures automatically induced on 4-dimensional moduli spaces of rational curves with normal bundle $N = \mathbb{C}^2 \otimes \mathcal{O}(1)$. In this section we show that moduli spaces of compact complex hypersurfaces often come equipped canonically with induced projective structures satisfying some natural integrability conditions. Other manifestations of general and strong links between complex analysis and di erential geometry can be found in Merkulov's survey [M].

Theorem 1. *Let $X \hookrightarrow Y$ be a compact complex submanifold of codimension 1 with normal bundle N such that $H^1(X, N) = 0$ and let M be the associated complete moduli space of relative deformations of X inside Y. If $H^1(X, \mathcal{O}_X) = 0$, then a sufficiently small neighbourhood $M_0 \subset M$ of the point $t_0 \in M$ corresponding to X, comes equipped canonically with a projective structure such that, for every point $y \in Y' \equiv \bigcup_{t \in M} X_t$, the associated submanifold $P_y \subseteq \nu \circ \mu^{-1}(y) \cap M_0$ is totally geodesic.*

Proof. An open neighbourhood of the submanifold $X \hookrightarrow Y$ can always be covered by a nite number of coordinate charts $\{W_i\}$ with local coordinate functions (w_i, z_i^a), $a = 1, \ldots, n = \dim X$, on each neighbourhood W_i such that $X \cap W_i$ coincides with the subspace of W_i determined by the equation $w_i = 0$. On the intersection $W_i \cap W_j$ the coordinates w_i, z_i^a are holomorphic functions of w_j and z_j^b,

$$w_i = f_{ij}(w_j, z_j^b), \qquad z_i^a = g_{ij}^a(w_j, z_j^b),$$

with $f_{ij}(0, z_j^b) = 0$. Here $z_j = (z_j^1, \ldots, z_j^n)$.

Let $U \subset M$ be a coordinate neighbourhood of the point t_0 with coordinate functions t^α, $\alpha = 1, \ldots, m = \dim M$. Then the coordinate domains $U \times W_i$ with

coordinate functions (w_i, z_i^a, t^α) cover an open neighbourhood of $X \times U$ in the manifold $Y \times U$. For a sufficiently small U, the submanifold $F_U \cong \nu^{-1}(U) \hookrightarrow Y \times U$ is described in each coordinate chart $W_i \times U$ by an equation of the form [K-1]

$$w_i = \phi_i(z_i, t),$$

where $\phi_i(z_i, t)$ is a holomorphic function of z_i^a and t^α which satisfies the boundary conditions $\phi_i(z_i, t_0) = 0$. For each fixed $t \in U$ this equation defines a submanifold $X_t \cap W_i \hookrightarrow W_i$.

By construction, F_U is covered by a finite number of coordinate neighbourhoods $\{V_i \equiv W_i \times U|_F\}$ with local coordinate functions (z_i^a, t^α) which are related to each other on the intersections $V_i \cap V_j$ as

$$z_i^a = g_{ij}^a(\phi_j(z_j, t), z_j).$$

Obviously we have $\phi_i(g_{ij}(\phi_j(z_j, t), z_j), t) = f_{ij}(\phi_j(z_j, t), z_j)$.

The Kodaira map $k : TM|_U \to \nu_*^0(N_F|_{F_U})$ can be described in the following way: take any vector field v on U and apply the corresponding 1st-order differential operator $V^\alpha \partial_\alpha$, where $\partial_\alpha = \partial/\partial t^\alpha$, to each function $\phi_i(z_i, t)$. The result is a collection of holomorphic functions $\sigma_i(z_i, t) = V^\alpha \partial_\alpha \phi_i(z_i, t)$ defined respectively on V_i. On the intersection $V_i \cap V_j$ one has $\sigma_i(z_i, t)|_{z_i = g_{ij}(\phi_j, z_j)} = F_{ij}(z_j, t)\sigma_j(z_j, t)$, where

$$F_{ij} \equiv \left.\frac{\partial f_{ij}}{\partial w_j}\right|_{w_j = \phi_j(z_j, t)} - \left.\frac{\partial \phi_i}{\partial z_i^a}\right|_{z_i = g_{ij}(\phi_j, z_j)} \left.\frac{\partial g_{ij}^a}{\partial w_j}\right|_{w_j = \phi_j(z_j, t)}$$

is the transition matrix of the normal bundle $N_F|_{F_U}$ on the overlap $F_U \cap V_i \cap V_j$. Therefore the 0-cochain $\{\sigma_i(z_i, t)\}$ is a Cech 0-cocycle representing a global section $k(v)$ of the normal bundle N_F over F_U.

Let us investigate how second partial derivatives of $\{\phi_i(z_i, t)\}$ and $\{\phi_j(z_j, t)\}$ are related on the intersection $V_i \cap V_j$. Since

$$\left.\frac{\partial \phi_i(z_i, t)}{\partial t^\alpha}\right|_{z_i = g_{ij}(\phi_j, z_j)} = F_{ij}\frac{\partial \phi_j(z_j, t)}{\partial t^\alpha}$$

we find

$$(2) \qquad \left.\frac{\partial^2 \phi_i}{\partial t^\alpha \partial t^\beta}\right|_{z_i = g_{ij}(\phi_j, z_j)} = F_{ij}\frac{\partial^2 \phi_j}{\partial t^\alpha \partial t^\beta} + E_{ij}\frac{\partial \phi_j}{\partial t^\alpha}\frac{\partial \phi_j}{\partial t^\beta} - G_{ij\alpha}\frac{\partial \phi_j}{\partial t^\beta} - G_{ij\beta}\frac{\partial \phi_j}{\partial t^\alpha},$$

where

$$E_{ij} = \left.\frac{\partial^2 f_{ij}}{\partial w_j \partial w_j}\right|_{w_j = \phi(z_j, t)} - \left.\frac{\partial \phi_i}{\partial z_i^a}\right|_{z_i = g_{ij}(\phi_j, z_j)} \left.\frac{\partial^2 g_{ij}^a}{\partial w_j \partial w_j}\right|_{w_j = \phi_j(z_j, t)}$$
$$- \left.\frac{\partial^2 \phi_i}{\partial z_i^a \partial z_i^b}\right|_{z_i = g_{ij}(\phi_j, z_j)} \left.\left(\frac{\partial g_{ij}^a}{\partial w_j}\frac{\partial g_{ij}^b}{\partial w_j}\right)\right|_{w_j = \phi_j(z_j, t)},$$

and

$$G_{ij\alpha} = \left.\frac{\partial^2 \phi_i}{\partial z_i^a \partial t^\alpha}\right|_{z_i = g_{ij}(\phi_j, z_j)} \left.\frac{\partial g_{ij}^a}{\partial w_j}\right|_{w_j = \phi_j(z_j, t)}.$$

The collections $\{E_{ij}\}$ and $\{G_{ij\alpha}\}$ form 1-cochains with coefficients in N_F^* and $\nu^*(\Omega^1 M)$, respectively. Straightforward calculations reveal the obstructions for

these two 1-cochains to be 1-cocycles,

$$\delta\{E_{ik}\} = 2\frac{\partial F_{ij}(z_j,t)}{\partial z_j^a}\left.\frac{\partial g_{jk}^a}{\partial w_k}\right|_{w_k=\phi_k(z_k,t)},$$

$$\delta\{G_{ik\alpha}\} = \frac{\partial F_{ij}(z_j,t)}{\partial z_j^a}\left.\frac{\partial g_{jk}^a}{\partial w_k}\right|_{w_k=\phi_k(z_k,t)}\frac{\partial\phi_j(z_j,t)}{\partial t^\alpha}.$$

From these equations we conclude that the 1-cochain $\{\tau_{ik\alpha}\}$, where

$$\tau_{ik\alpha} \equiv \frac{1}{2}E_{ik}\frac{\partial\phi_k}{\partial t^\alpha} - G_{ik\alpha},$$

is actually a 1-cocycle with values in $\nu^*(\ ^1M)$. Since $H^1(X,\mathcal{O}_X) = 0$, the semi-continuity principle [K-2] implies $H^1(X_t,\mathcal{O}_{X_t}) = 0$ for all points in some Stein neighbourhood $M_0 \subseteq U$. Hence, by the Leray spectral sequence

$$H^1(\nu^{-1}(M_0),\nu^*(\ ^1M)) = 0.$$

Therefore, the 1-cocycle $\{\tau_{ik\alpha}\}$ is always a coboundary $\{\tau_{ij\alpha}\} = \delta\{\theta_{i\alpha}\}$, or more explicitly,

(3) $$\tau_{ij\alpha}(z_j,t) = F_{ij}(z_j,t)(-\theta_{i\alpha}(z_i,t)|_{z_i=g_{ij}(\phi_j,z_j)} + \theta_{j\alpha}(z_j,t)),$$

for some 0-cochain $\{\theta_{i\alpha}(z_i,t)\}$ on $\nu^{-1}(M_0)$ with values in $\nu^*(\ ^1M)$. However, this 0-cochain is de ned non-uniquely| for any global section $\xi = \xi_\alpha dt^\alpha$ of $\nu^*(\ ^1M)$ over $\nu^{-1}(M_0)$ the 0-cochain

(4) $$\theta_{i\alpha}(z_i,t) = \theta_{i\alpha}(z_i,t) + \xi_\alpha(t)|_{\nu^{-1}(M_0)\cap V_i}$$

splits the same 1-cocycle $\{\tau_{ij\alpha}\}$. Note that, due to the compactness of the complex submanifolds $\nu^{-1}(t) \subset F$ for all $t \in M_0$ the components ξ_α of the global section $\xi \in H^0(\nu^{-1}(M_0),\nu^*(\ ^1M))$ are constant along the bers, i.e. $\xi_\alpha \in \nu^{-1}(\mathcal{O}_{M_0})$.

If we rewrite equation (2) in the form

$$\left.\frac{\partial^2\phi_i(z_i,t)}{\partial t^\alpha\partial^\beta}\right|_{z_i=g_{ij}(\phi_j,z_j)} = F_{ij}(z_j,t)\frac{\partial^2\phi_j(z_j,t)}{\partial t^\alpha\partial t^\beta}$$

$$+ \tau_{ij\alpha}(z_j,t)\frac{\partial\phi_j(z_j,t)}{\partial t^\beta} + \tau_{ij\beta}(z_j,t)\frac{\partial\phi_j(z_j,t)}{\partial t^\alpha}$$

and take equation (3) into account, we obtain the equality

$$\left.\left(\frac{\partial^2\phi_i}{\partial t^\alpha\partial t^\beta} + \theta_{i\alpha}\frac{\partial\phi_i}{\partial t^\beta} + \theta_{i\beta}\frac{\partial\phi_i}{\partial t^\alpha}\right)\right|_{z_i=g_{ij}(\phi_j,z_j)} = \frac{\partial^2\phi_j}{\partial t^\alpha\partial t^\beta} + \theta_{j\alpha}\frac{\partial\phi_j}{\partial t^\beta} + \theta_{j\beta}\frac{\partial\phi_j}{\partial t^\alpha}$$

which implies that, for each value of α and β, the holomorphic functions,

$$_{i\alpha\beta}(z_i,t) \equiv \frac{\partial^2\phi_i(z_i,t)}{\partial t^\alpha\partial t^\beta} + \theta_{i\alpha}(z_i,t)\frac{\partial\phi_i(z_i,t)}{\partial t^\beta} + \theta_{i\beta}(z_i,t)\frac{\partial\phi_i(z_i,t)}{\partial t^\alpha},$$

represent a global section of the normal bundle N_F over $\nu^{-1}(M_0)$. Since the collections of functions $\{\partial_\alpha\phi_i(z_i,t)\}$ form a Cech representation of a basis for the free $\mathcal{O}_{M_0}$-module $\nu_*^0(N_F|_{\nu^{-1}(M_0)})$, the equality

(5) $$_{i\alpha\beta}(z_i,t) =\ _{\alpha\beta}^\gamma\partial_\alpha\phi_i(z_i,t)$$

must hold for some global holomorphic functions $_{\alpha\beta}^\gamma$ on $\nu^{-1}(M_0)$. Since all the bers $\nu^{-1}(t)$, $t \in M_0$, are compact complex manifolds, these functions are actually pull-backs of some holomorphic functions on M_0. A coordinate system $\{t^\alpha\}$ on M_0 was used in the construction of $_{\alpha\beta}^\gamma(t)$. However from (5) it immediately follows

that under general coordinate transformations $t^\alpha \to t^{\alpha'} = t^{\alpha'}(t^\beta)$ these functions transform according to

$$\gamma'_{\alpha'\beta'} = \frac{\partial t^{\gamma'}}{\partial t^\delta} \left(\delta_{\mu\nu} \frac{\partial t^\mu}{\partial t^{\alpha'}} \frac{\partial t^\nu}{\partial t^{\beta'}} + \frac{\partial^2 t^\delta}{\partial t^{\alpha'} \partial t^{\beta'}} \right).$$

Thus from any given splitting $\{\tau_{ij\alpha}\} = \delta\{\theta_{i\alpha}\}$ of the 1-cocycle $\{\tau_{ij\alpha}\}$ we extract a symmetric affine connection $\gamma_{\alpha\beta}(t)$. It is straightforward to check that this connection is independent of the choice of the (w_i, z_i^a)-coordinate system used in the construction and thus is well-defined except for the arbitrariness in its construction described by the transformations (4) which, as one can easily check, change the connection as follows:

$$\theta_{i\alpha}(z_i, t) \to \theta_{i\alpha}(z_i, t) + \xi_\alpha(t)$$
$$\gamma_{\alpha\beta}(t) \to \gamma_{\alpha\beta}(t) + \xi_\alpha(t)\delta^\gamma_\beta + \xi_\beta(t)\delta^\gamma_\alpha.$$

Therefore we conclude that the neighbourhood M_0 of the point t_0 in the moduli space comes equipped *canonically* with a projective structure.

Let us now prove that for each point $y_0 \in Y' = \bigcup_{t \in M_0} X_t$, the associated submanifold $P_y \subseteq \nu \circ \mu^{-1}(y) \subset M_0$ is totally geodesic relative to the canonical projective connection in M_0. Suppose that $y_0 \in W_i$ for some i. Then $y_0 = (w_{i0}, z_{i0}^a)$ and the submanifold P_{y_0} is given locally by the equations $w_{i0} - \phi_i(z_{i0}, t) = 0$, where $t \in \nu \circ \mu^{-1}(y_0) \setminus \{\text{singular points}\}$. Then a vector field $v(t) = V^\alpha \partial_\alpha|_{P_{y_0}}$ is tangent to P_{y_0} if and only if it satisfies the simultaneous equations

$$(6) \qquad V^\alpha \partial_\alpha \phi_i(z_{i0}, t) = 0.$$

In order to prove that the submanifold P_{y_0} for arbitrary $y_0 \in Y'$ is totally geodesic relative to the canonical projective connection, we have to show that, for any vector fields $v(t) = V^\alpha \partial_\alpha$ and $w(t) = W^\alpha \partial_\alpha$ on P_{y_0}, the equation

$$(7) \qquad (W^\beta \partial_\beta V^\alpha + {}^\alpha_{\beta\gamma} V^\gamma W^\beta) \bmod TP_{y_0} = 0$$

holds. Since $v(t)$ and $w(t)$ are tangent to $P_{y_0} \subset M$, we have the equation

$$(8) \qquad W^\beta(t) \frac{\partial}{\partial t^\beta} (V^\alpha \partial_\alpha \phi_i(z_{i0}, t)) = 0,$$

$$V^\alpha W^\beta \frac{\partial^2 \phi_i(z_{i0}, t)}{\partial t^\alpha \partial t^\beta} = V^\alpha W^\beta {}^\gamma_{\alpha\beta} \frac{\partial \phi_i(z_{i0}, t)}{\partial t^\gamma}.$$

From the latter equation and equation (8) it follows that

$$(W^\beta \partial_\beta V^\alpha + {}^\alpha_{\beta\gamma} V^\gamma W^\beta) \frac{\partial \phi_i(z_{i0}, t)}{\partial t^\alpha} = 0.$$

By (6) this means that $(W^\beta \partial_\beta V^\alpha + {}^\alpha_{\beta\gamma} V^\gamma W^\beta) \partial_\alpha \in TP_{y_0}$, and thus equation (7) holds. The proof is completed. $\square$

We may have a moduli space even if the condition $H^1(X, N) = 0$ is not satisfied. Given a moduli space, the proof above provides a projective structure so we have the following global result.

Corollary 2. *Let $\{X_t \hookrightarrow Y | t \in M\}$ be a complete analytic family of compact complex hypersurfaces such that $H^1(X_t, \mathcal{O}_{X_t}) = 0$ for all $t \in M$. Then the moduli space M comes equipped canonically with a projective structure such that, for every point $y \in Y'$, the associated submanifold $P_y = \nu \circ \mu^{-1}(y) \subset M$ is totally geodesic.*

We conclude this section with a brief geometric interpretation of geodesics canonically induced on moduli spaces of compact complex hypersurfaces. Any complex curve (immersed connected complex 1-manifold) in a complex manifold M has a canonical lift to a complex curve in the projectivized tangent bundle $P_M(TM)$ | one simply associates to each point of the curve its tangent direction. Then a projective structure on M de nes a family of lifted curves in $P_M(TM)$ which foliates the projectivized bundle holomorphically [H, L]. Then, for geodesically convex M, the quotient space of this foliation, Z, is a $(2n-2)$-dimensional manifold, where $n = \dim M$. There is a double bration

$$\text{(9)} \qquad\qquad\qquad Z \xleftarrow{\tau} P_M(TM) \xrightarrow{\sigma} M$$

such that, for each $z \in Z, \sigma \circ \tau^{-1}(z) \subset M$ is a geodesic from the projective structure; for each $t \in M$, $\tau \circ \sigma^{-1}(t) \subset Z$ is projective space $\mathbb{CP}^{n-1}$ embedded into Z with normal bundle $T\mathbb{CP}^{n-1}(-1)$ [L].

Let $X_0 \hookrightarrow Y$ be a compact complex submanifold of codimension 1 such that $H^1(X_0, N) = H^1(X_0, \mathcal{O}_{X_0}) = 0$ and let M be a geodesically convex domain in the associated complete moduli space of relative deformations of X_0 inside Y. The space of geodesics Z can be identi ed in this case with the family of intersections $X_s \cap X_t \subset Y$, $s, t \in M$. From the explicit coordinate description of submanifolds $X_t \subset Y$ given in the proof of Theorem 1 one can easily see that, for each $t \in M$, the intersection $X_t \cap X_0$ is a divisor of the holomorphic line bundle on X_0 which is a holomorphic deformation of the normal bundle N. Since $H^1(X_0, \mathcal{O}_X) = 0$, any holomorphic deformation of N must be isomorphic to N [K-3]. Therefore, each intersection $X_t \cap X_0$ is a divisor of the normal bundle on X_0, and, by completeness of the family $\{X_t \hookrightarrow Y | t \in M\}$, all divisors of N arise in this way. If $t_0 \in M$ is the point associated to $X_0 \subset Y$ via the double bration (1), then the set of all intersections $X_0 \cap X_t$ is a projective space $\mathbb{CP}^{\dim M - 1} \subset Z$ associated to t_0 via the double bration (9). Then a geodesic through the point $t_0 \in M$ is a family of X_t which have the same intersection with X_0.

4. Applications and examples

One of the immediate applications of the theorem on projective connections is in the theory of 3-dimensional Einstein-Weyl manifolds. Hitchin [H] proved that there is a one-to-one correspondence between local solutions of Einstein-Weyl equations in 3 dimensions and pairs (X, Y), where Y is a complex 2-fold and X is the projective line $\mathbb{CP}^1$ embedded into Y with normal bundle $N \simeq \mathcal{O}(2)$. However the corresponding twistor techniques allowed one to compute only part of the canonical Einstein-Weyl structure induced on the complete moduli space M of relative deformations of X in Y, namely the conformal structure on M. Although the geodesics were formally described and the existence of a connection with special curvature was proved, no explicit formula for the connection was obtained. The theorem on projective connections lls this gap and provides one with a technique which is capable of decoding the full Einstein-Weyl structure from the holomorphic data of the embedding $X \hookrightarrow Y$. We shall use Theorem 1 in some examples to compute explicitly the canonical projective connection and then the canonical Einstein-Weyl structure on the complete moduli space of rational curves embedded into a 2-dimensional complex manifold with normal bundle $N \simeq \mathcal{O}(2)$.

Consider a non-singular curve X of bidegree $(1, n)$ in the quadric $\mathbb{CP}^1 \times \mathbb{CP}^1$. Then X is rational and has normal bundle $\mathcal{O}(2n)$ [P]. The space M of such curves can be described as follows: Let (ζ, η) be affine coordinates on $\mathbb{CP}^1 \times \mathbb{CP}^1$ and consider the graph of a rational function of degree n:

$$\eta = \frac{P(\zeta)}{Q(\zeta)},$$

(10)

$$P(\zeta) = a_n \zeta^n + a_{n-1} \zeta^{n-1} + \cdots + a_0,$$

$$Q(\zeta) = b_n \zeta^n + b_{n-1} \zeta^{n-1} + \cdots + b_0.$$

The family of such $(1, n)$-curves is parameterized by $\mathbb{CP}^{2n+1}$ and the space M of non-singular curves is $\mathbb{CP}^{2n+1} \setminus R$ where R is the manifold of codimension 1 and degree $2n$ given by the resultant of P and Q. The geodesics of the projective connection are again given by projective lines in $\mathbb{CP}^{2n+1} \setminus R$. We may of course choose to describe the induced structure on the hypersurface given by $R = 1$, and for $n = 1$ this corresponds to the standard projective structure on $SL(2, \mathbb{C})$ or on one of its real slices $\mathcal{H}^3, S^3$.

In order to obtain less trivial examples we consider branched coverings. Consider a complex curve C contained in a complex surface S. We want to construct a branched covering of a neighbourhood of C branched along C. Choose coordinates (x_i, y_i) on neighbourhoods O_i along C such that $O_i \cap C$ is given by $x_i = 0$. Then on overlaps we have $x_i = x_j H_{ij}(x_j, y_j)$ and $y_i = K_{ij}(x_j, y_j)$. Now, we look for an n-fold cover branched along C: take patches W_i with coordinates (w_i, z_i) and define the covering map $(w_i, z_i) \to (x_i, y_i) = (w_i^n, z_i)$. This is a branched cover of O_i branched along $O_i \cap C$. We want to identify the neighbourhoods W_i along the curve C to obtain a surface Y with a map $\pi : Y \to S$ which locally has the form above. We get

$$w_i^n = x_i = x_j H_{ij}(z_j, y_j) = w_j^n H_{ij}(w_j^n, z_j).$$

If we make a choice of the n-th root and put $\widetilde{H}_{ij} = H_{ij}^{1/n}$ we get

$$w_i = w_j \widetilde{H}_{ij}(w_j^n, z_j) = f_{ij}(w_j, z_j).$$

The obstruction for this to work along the curve is the class $\widetilde{H}_{ij} \widetilde{H}_{jk} \widetilde{H}_{ki} \in H^2(C, \mathbb{Z}/n)$. We can identify this obstruction to be the self-intersection number of C modulo n: since $dx_i = H_{ij}(0, y_j) dx_j$ we see that $H_{ij}(0, y_j)$ represents the normal bundle N in $H^1(C, \mathcal{O}^*)$. From the long exact sequence associated with

$$0 \to \mathbb{Z} \to \mathcal{O} \overset{\exp}{\to} \mathcal{O}^* \to 0$$

we see that the degree of N is equal to $\log H_{ij} + \log H_{jk} + \log H_{ki}$. Thus, the obstruction to obtain Y is equal to the self-intersection of C, modulo n. Each choice of $H_{ij}^{1/n}$ corresponds to an element in $H^1(C, \mathbb{Z}/n)$. Unless the homology class of C in $H_2(S, \mathbb{Z})$ is divisible by n this local construction along the curve cannot be extended to work globally on S [A].

Now, let us return to the case where C is a $(1, n)$-curve in $\mathbb{CP}^1 \times \mathbb{CP}^1$. In this case $C \cong \mathbb{CP}^1$, so there is a unique n-fold covering Y branched along C which we cannot extend to all of $\mathbb{CP}^1 \times \mathbb{CP}^1$. The branch locus $X \subseteq Y$ is a copy of C but $\deg N_X = \frac{1}{n} \deg N_C = 2$ so we may describe an Einstein-Weyl structure on the moduli space of curves in Y [H] and contrary to earlier attempts we are now able to get the connection $\overset{\gamma}{}_{\alpha\beta}$ explicitly. Let us concentrate on $(1, 2)$ curves and let C be

the curve $\eta = \zeta^2$. The projection π maps the curves in Y onto those $(1,2)$-curves which meet C in two points to second order. These curves may be given as in (10) with

$$P(\zeta) = \zeta^2 - 2t_0 t_1 \zeta - t_0^2,$$
$$Q(\zeta) = t_2^2 \zeta^2 + 2t_1 t_2 \zeta + 1 + 2t_0 t_2 + t_1^2$$

(see [P]). In order to describe the lifted curves we introduce the coordinates

$$x_1 = \eta - \zeta^2, \qquad x_2 = \tilde{\eta} - \tilde{\zeta}^2,$$
$$y_1 = \zeta, \qquad\qquad y_2 = \tilde{\zeta},$$

where $(\tilde{\zeta}, \tilde{\eta}) = (\frac{1}{\zeta}, \frac{1}{\eta})$. Then C is given by $x_i = 0$. Making the coordinate transformation

$$(x_1, y_1) \rightarrow (w, z) = (\sqrt{x_1}, y_1),$$
$$(x_2, y_2) \rightarrow (\tilde{w}, \tilde{z}) = (\sqrt{x_1}, y_1)$$

we arrive at a covering of Y by two coordinate charts W and $\widehat{W}$ which is exactly of the type used in the proof of Theorem 1 and has the transition functions

$$\tilde{w} = f(w, z), \qquad \tilde{z} = g(z),$$

given by

$$f(w, z) = \frac{w}{z\sqrt{w^2 + z^2}}, \qquad g(z) = z^{-1}.$$

The complete maximal family of relative deformations of C is described in this chart by the equations (in the notation of the proof of Theorem 1) $w = \phi(z, t)$ and $\tilde{w} = \hat{\phi}(\tilde{z}, t)$, with

$$\phi(z, t) = iR(z)Q(z)^{-1/2}, \qquad \hat{\phi}(z, t) = iR(z)P(z)^{-1/2},$$

where $R(z) = t_2 z^2 + t_1 z + t_0$. Note that a useful identity $P = z^2 Q - R^2$ holds [P].

Now we have all the data to apply the machinery developed in the proof of Theorem 1. Following that scenario one finds that the canonical projective structure on M can be represented by the following torsion-free affine connection

$$\Gamma_{01}^0 = t_1(1 + 3t_0 t_2)(2\Delta)^{-1}, \qquad\qquad \Gamma_{01}^1 = t_2(2 + t_1^2 + 2t_0 t_2)(2\Delta)^{-1},$$
$$\Gamma_{00}^0 = t_2(1 + t_0 t_2)\Delta^{-1}, \qquad\qquad \Gamma_{00}^1 = -t_1 t_2^2 \Delta^{-1},$$
$$\Gamma_{02}^0 = t_0(1 + t_0 t_2 + t_1^2)(2\Delta)^{-1}, \qquad \Gamma_{02}^1 = -t_1(1 + t_1^2)(2\Delta)^{-1},$$
$$\Gamma_{11}^0 = -t_0(1 + t_0 t_2)\Delta^{-1}, \qquad\qquad \Gamma_{11}^1 = t_0 t_1 t_2 \Delta^{-1},$$
$$\Gamma_{12}^0 = -t_0^2 t_1(2\Delta)^{-1}, \qquad\qquad\quad \Gamma_{12}^2 = -t_0(1 + t_0 t_2 + t_1^2)(2\Delta)^{-1}$$

and all other Christoffel symbols being zero. Here

$$\Delta = (1 + t_0 t_2)^2 + t_1^2(1 + 2t_0 t_2).$$

Note that $\Delta^2 = R$ where R is the resultant of the polynomials in (10).

The conformal structure $[g]$ on M is given by the condition for the curves to meet to second order. Thus we may choose the following metric in the conformal structure [P]

(11)
$$g = t_1^2 t_2^2 dt_0^2 + (1 + t_0 t_2)^2 dt_1^2 + 4t_0^2(1 + t_1^2)dt_2^2 + 2t_1 t_2(1 + t_0 t_2)dt_0 dt_1$$
$$- 4(1 + t_1^2)(1 + t_0 t_2)dt_0 dt_2 - 4t_0^2 t_1 t_2 dt_1 dt_2.$$

Since our connection ∇ is projectively equivalent to the Weyl connection D it satisfies

$$(\nabla g)_{\alpha\beta\gamma} = a_\alpha g_{\beta\gamma} + b_\beta g_{\alpha\gamma} + b_\gamma g_{\alpha\beta}$$

for some 1-forms $a = \sum_{i=0}^{2} a_\alpha dt^\alpha$ and $b = \sum_{i=0}^{2} b_\alpha dt^\alpha$. We may solve these equations and present the Weyl connection D in terms of the Levi-Civita connection ∇^g and the 1-form $\omega = a - 2b = \sum_\alpha \omega_\alpha dt_\alpha$,

$$D = \nabla^g + \frac{1}{2}\omega^\# g - \omega \odot I;$$

see [PT]. We get

$$a_0 = 3t_1^2 t_2 (2\ )^{-1}, \qquad\qquad b_0 = -3t_1^2 t_2 (4\ )^{-1},$$
$$a_1 = -3t_1(1 + t_0 t_2)(4\ )^{-1}, \qquad b_1 = -3t_1(1 + t_0 t_1)(4\ )^{-1},$$
$$a_2 = -3t_0(1 + t_0 t_2 + t_1^2)(2\ )^{-1}, \qquad b_2 = -3t_0(1 + t_0 t_2 + t_1^2)(2\ )^{-1}.$$

Thus using only the methods of the relative deformation theory of compact hypersurfaces we computed the full Einstein-Weyl structure on the moduli space.

Suppose we blow up a point s on the quadric and take a $(1, n)$-curve passing through the point. Then in the blown up surface the curve will have self-intersection number $2n - 1$ and this corresponds to considering all the $(1, n)$-curves passing through s. We may combine this with the branched covering construction. In [PT] we considered the Einstein-Weyl structure associated to the $(1, 3)$-curves: first we considered the 2-fold branched cover which reduced the degree of the normal bundle from 6 to 3 and then we blew up a point on the branch locus to get self-intersection equal to 2. Again we may compute the Weyl connection or compute the connection associated to any combination of blow up and branched cover. This will give non trivial examples with normal bundle $\mathcal{O}(n)$ for any n.

Acknowledgments

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