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STABILITY OF THE LOCAL SPECTRUM

TERESA BERMÚDEZ, MANUEL GONZÁLEZ, AND ANTONIO MARTINÓN

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Abstract. We give some conditions implying the equality of local spectra

$$\sigma(x,T)=\sigma(f[T]x,T),$$

where $T: X \longrightarrow X$ is a (bounded linear) operator on a complex Banach space X, and $f[T]x$ is defined by means of a local functional calculus. Moreover, we give conditions implying the stability of the local spectrum for the holomorphic and the meromorphic functional calculi.

1. Introduction

Let X be a complex Banach space and let T be a (bounded linear) operator defined on X. For every $x \in X$, the operator T has a local spectrum $\sigma(x,T)$ which is a useful tool in the study of the structure of the spectrum and the invariant subspaces of T.

The problem we address is the detection of vectors y which have the same local spectrum as a fixed vector x, namely

$$\sigma(x,T)=\sigma(y,T). \tag{1}$$

This problem has deserved the attention of several authors. In [2], Erdelyi and Lange prove that if T is an operator satisfying the Single Valued Extension Property (hereafter referred to as SVEP) and $\widehat{x}_T$ is the local resolvent function of T in x, then

$$\sigma(\widehat{x}_T(\lambda),T)=\sigma(x,T) \tag{2}$$

for all $\lambda \in \mathbb{C}\setminus\sigma(x,T)$. Moreover, if A is an operator which commutes with an operator T satisfying the SVEP, then

$$\sigma(Ax,T)\subset\sigma(x,T), \tag{3}$$

for all $x \in X$. In particular, if A has an inverse, then the expression (3) turns into an equality. It also follows, from the results derived by Bartle [1], that given $\lambda \in \mathbb{C}$ and $n \in \mathbb{N}$, we have

$$\sigma((\lambda-T)^n x,T)\subset\sigma(x,T)\subset\sigma((\lambda-T)^n x,T)\cup\{\lambda\}. \tag{4}$$

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Hence if $\lambda \notin \sigma(x,T)$, then

$$\sigma(x,T) = \sigma((\lambda - T)^n x, T).$$

Finally McGuire [7] shows that if T is an operator in a complex separable Hilbert space H with an empty point spectrum, and f is an analytic function on an open set $\triangle(f)$ containing $\sigma(x,T)$, not identically zero on any component of $\triangle(f)$, then

$$\sigma(f[T]x, T) = \sigma(x,T),$$

where $f[T]x$ is defined by using the "Cauchy formula" with the local resolvent of T in x (see below).

In this paper we give conditions implying the equality

$$\sigma(x,T) = \sigma(Ax,T)$$

for certain operators A obtained from T using the meromorphic functional calculus or the local functional calculus. Our results include those of [1], [2] and [7].

2. Preliminaries

Let X be a complex Banach space. We denote by $L(X)$ the class of all (bounded linear) operators on X, and by $C(X)$ the class of all closed operators T with *domain* $D(T)$ and *range* $R(T)$ in X.

Given $T \in C(X)$, we have that λ belongs to $\rho(T)$, the *resolvent set* of T, if there exists $(\lambda - T)^{-1} \in L(X)$ such that $R((\lambda - T)^{-1}) = D(T)$ and for every $x \in X$ we have

$$(\lambda - T)(\lambda - T)^{-1}x = x.$$

We denote by $\sigma(T) := \mathbb{C} \setminus \rho(T)$ the *spectrum set* of T. Note that the set $\rho(T)$ is open and the *resolvent function* $\lambda \longrightarrow (\lambda - T)^{-1}$ is analytic in $\rho(T)$.

Likewise, for every $x \in X$ the local spectral theory is defined as follows. We say that $\lambda \in \rho(x,T)$, the *local resolvent set* of T in x, if there exists an analytic function $w : U \longrightarrow X$ defined on a neighborhood U of λ, which satisfies the equation

$$(\mu - T)w(\mu) = x,$$

for every $\mu \in U$. We denote by $\sigma(x,T) := \mathbb{C} \setminus \rho(x,T)$ the *local spectrum set* of T in x. Since w is not necessarily unique, a property is introduced to avoid this problem.

A closed linear operator $T : D(T) \subset X \longrightarrow X$ satisfies the SVEP if for every analytic function $h : \triangle(h) \longrightarrow X$ defined on an open set $\triangle(h) \subset \mathbb{C}$, the condition $(\lambda - T)h(\lambda) \equiv 0$ implies $h \equiv 0$. If T satisfies the SVEP, then for every $x \in X$ there exists a unique maximal analytic function $\widehat{x}_T : \rho(x,T) \longrightarrow X$ such that

$$(\lambda I - T)\widehat{x}_T(\lambda) = x,$$

for every $\lambda \in \rho(x,T)$. The function $\widehat{x}_T$ is called the *local resolvent function* of T at x. See [2], [3] and [6] for further details.

In the following proposition we recall from [2] some basic properties for operators satisfying the SVEP.

Proposition 1. *Let $T \in L(X)$ satisfy the SVEP and let $x \in X$. Then the following assertions hold:*

(i) *If $\lambda \in \rho(x,T)$, then $\sigma(\widehat{x}_T(\lambda), T) = \sigma(x,T)$.*

(ii) *If $S \in L(X)$ commutes with T and $y = Sx$, then $S\widehat{x}_T(\lambda) = \widehat{y}_T(\lambda)$ for $\lambda \in \rho(x, T)$. In particular, $\sigma(Sx, T) \subset \sigma(x, T)$.*

For $T \in L(X)$, the *holomorphic functional calculus* is defined as follows [9]. Let f be an analytic function defined on an open set $\triangle(f)$ containing $\sigma(T)$. The operator $f(T) \in L(X)$ is defined by the "Cauchy formula"

$$f(T) := \frac{1}{2\pi i}\int_\Gamma f(\lambda)R(\lambda, T)d\lambda,$$

where Γ is the boundary of a Cauchy domain D such that $\sigma(T) \subset D \subset \triangle(f)$.

This definition may be extended to meromorphic functions. Let f be a meromorphic function in an open set $\triangle(f)$ containing $\sigma(T)$, such that the poles of f are not in the point spectrum $\sigma_p(T)$, and let $\alpha_1, \dots, \alpha_k$ be the poles of f in $\sigma(T)$, with multiplicity $n_1, \dots, n_k$, respectively. We consider the polynomial p given by

$$p(\lambda) = \prod_{i=1}^{k}(\alpha_i - \lambda)^{n_i}.$$

Note that $g(\lambda) := f(\lambda)p(\lambda)$ is an analytic function. In [4], Gindler defines a *meromorphic functional calculus* by

$$f\{T\} := g(T)p(T)^{-1}.$$

In this way he obtains an operator $f\{T\} \in C(X)$. Clearly, the meromorphic calculus is an extension of the holomorphic calculus.

3. The local functional calculus

Let f be an analytic function defined on an open set $\triangle(f)$. For H a Hilbert space and $T \in L(H)$ an operator with empty point spectrum, McGuire [7] introduces a *local functional calculus* in which he defines $f[T]x$, for $x \in H$ with $\sigma(x, T) \subset \triangle(f)$, by

$$f[T]x = \frac{1}{2\pi i}\int_\Gamma f(\lambda)\widehat{x}_T(\lambda)d\lambda, \tag{5}$$

where Γ is the boundary of a Cauchy domain D such that $\sigma(x, T) \subset D \subset \triangle(f)$.

Using this idea, for any $T \in L(X)$ satisfying the SVEP we define an operator

$$f[T] : D(f[T]) \subset X \longrightarrow X$$

with domain

$$D(f[T]) := \{x \in X \ : \ \sigma(x, T) \subset \triangle(f)\}$$

and $f[T]x$ given by (5) for $x \in D(f[T])$.

Proposition 2. *Let $T \in L(X)$ satisfy the SVEP and let f be an analytic function in $\triangle(f)$. Then $D(f[T])$ is a linear subspace of X and $f[T]$ is a linear operator.*

Proof. It follows from $\sigma(x + y, T) \subset \sigma(x, T) \cup \sigma(y, T)$ [2, Proposition 1.5] and the definition. □

Next, we prove some results concerning the local functional calculus.

Proposition 3. *Let $T \in L(X)$ satisfy the SVEP and let f be an analytic function in $\triangle(f)$. Then the following assertions hold:*

(i) *If $S \in L(X)$ commutes with T, then S commutes with $f[T]$; i.e., $SD(f[T]) \subset D(f[T])$ and $Sf[T]x = f[T]Sx$ for all $x \in D(f[T])$.*

(ii) *If $x \in D(f[T])$ and $y := f[T]x$, then $f[T]\widehat{x}_T = \widehat{y}_T$ in $\rho(x,T)$, hence $\sigma(f[T]x,T) \subset \sigma(x,T)$.*

Proof. (i) Let $x \in D(f[T])$. By Proposition 1 we have $\sigma(Sx,T) \subset \sigma(x,T)$, and so $Sx \in D(f[T])$. Moreover, $S\widehat{x}_T$ is a restriction of the local resolvent of Sx. Then we have

$$\begin{aligned} Sf[T]x &= S\left(\frac{1}{2\pi i}\int_\Gamma f(\lambda)\widehat{x}_T(\lambda)d\lambda\right) \\ &= \frac{1}{2\pi i}\int_\Gamma f(\lambda)S\widehat{x}_T(\lambda)d\lambda \\ &= f[T]Sx. \end{aligned}$$

(ii) We prove that $f[T]\widehat{x}_T$ is analytic at every point $\lambda \in \rho(x,T)$. By Proposition 1 and using the expression for the local resolvent of $\widehat{x}_T(\lambda)$ in [2, Proposition 1.5] we have

$$\begin{aligned} f[T]\widehat{x}_T(\lambda) &= \frac{1}{2\pi i}\int_\Gamma f(\mu)\left(\frac{\widehat{x}_T(\mu)-\widehat{x}_T(\lambda)}{\lambda-\mu}\right)d\mu \\ &= \frac{1}{2\pi i}\int_\Gamma \frac{f(\mu)\widehat{x}_T(\mu)}{\lambda-\mu}d\mu - \frac{1}{2\pi i}\int_\Gamma \frac{f(\mu)\widehat{x}_T(\lambda)}{\lambda-\mu}d\mu. \end{aligned}$$

For every $x^* \in X^*$, the first integral satisfies

$$x^*\left(\frac{1}{2\pi i}\int_\Gamma \frac{f(\mu)\widehat{x}_T(\mu)}{\lambda-\mu}d\mu\right) = \frac{1}{2\pi i}\int_\Gamma \frac{f(\mu)}{\lambda-\mu}x^*(\widehat{x}_T(\mu))d\mu.$$

So it is analytic by [8, Theorem 10.7]. The second integral is analytic by Cauchy's Theorem. Consequently $f[T]\widehat{x}_T$ is analytic in $\rho(x,T)$. □

Assume $T \in L(X)$ satisfies the SVEP. Let $x \in X$, and let f,g be analytic functions such that $x \in D(f[T]) \cap D(g[T])$. Clearly we have $(\alpha f + \beta g)[T]x = \alpha f[T]x + \beta g[T]x$, for all $\alpha,\beta \in \mathbb{C}$.

Proposition 4. *Assume $T \in L(X)$ satisfies the SVEP. Let $x \in X$ and let f,g be analytic functions in a neighborhood of $\sigma(x,T)$. Then*

$$(fg)[T]x = f[T]g[T]x = g[T]f[T]x. \tag{6}$$

Proof. The proof is similar to that of the corresponding result for the holomorphic functional calculus [9].

First, taking into account part (ii) of Proposition 3, we obtain $\sigma(f[T]x,T) \subset \sigma(x,T) \subset \triangle(g)$, hence $f[T]x \in D(g[T])$, and analogously $g[T]x \in D(f[T])$. Moreover $\widehat{g[T]x}_T = g[T]\widehat{x}_T$.

Now if D_1, D_2 are bounded Cauchy domains such that $\sigma(x,T) \subset D_1$, $\overline{D_1} \subset D_2$ and $\overline{D_2} \subset \triangle(f) \cap \triangle(g)$, then we express $f[T](g[T]x)$ as an integral with respect to λ over Γ_1. Moreover we can write

$$f[T]g[T]x = \frac{1}{2\pi i}\int_{\Gamma_1} f(\lambda)(\widehat{g[T]x})_T(\lambda)d\lambda = \frac{1}{2\pi i}\int_{\Gamma_1} f(\lambda)g[T]\widehat{x}_T(\lambda)d\lambda.$$

Now expressing $g[T]\widehat{x}_T(\lambda)$ as an integral over Γ_2 and using the expression for the local resolvent of $\widehat{x}_T(\lambda)$ in [2, Proposition 1.5], we obtain

$$f[T]g[T]x = \frac{1}{2\pi i}\int_{\Gamma_1} f(\lambda)\left\{\frac{1}{2\pi i}\int_{\Gamma_2} g(\eta)\left(\frac{\widehat{x}_T(\lambda)-\widehat{x}_T(\eta)}{\eta-\lambda}\right)d\eta\right\}d\lambda$$

$$= -\frac{1}{4\pi^2}\int_{\Gamma_1} f(\lambda)\widehat{x}_T(\lambda)\int_{\Gamma_2}\frac{g(\eta)}{\eta-\lambda}d\eta d\lambda + \frac{1}{4\pi^2}\int_{\Gamma_1} f(\lambda)\int_{\Gamma_2}\frac{\widehat{x}_T(\eta)g(\eta)}{\eta-\lambda}d\eta d\lambda.$$

Since $\lambda \in D_2$ and $\eta \notin \overline{D_1}$, we have

$$\frac{1}{2\pi i}\int_{\Gamma_1}\frac{g(\eta)}{\eta-\lambda}d\eta = g(\lambda), \qquad \frac{1}{2\pi i}\int_{\Gamma_1}\frac{f(\lambda)}{\eta-\lambda}d\lambda = 0.$$

Hence we obtain the desired result

$$f[T]g[T]x = \frac{1}{2\pi i}\int_{\Gamma_1} f(\lambda)g(\lambda)\widehat{x}_T(\lambda)d\lambda = (fg)[T]x.$$

□

Remark 1. Sometimes the results of evaluating $f[T]g[T]x$ and $(fg)[T]x$ are different, as it is shown by the following example: Let T be the operator on the Hilbert space $\ell_2(\mathbb{N})$ defined by $T(x_n) = (\frac{1}{n}x_n)$. It is clear that T satisfies the SVEP, since T is a self-adjoint operator. For $x = (x_n) \in \ell_2(\mathbb{N})$ we have

$$\sigma(x,T) = \overline{\left\{\frac{1}{n} \;:\; x_n \neq 0\right\}}.$$

We take $x =: (1,1,0,\dots)$, $f(\lambda) := \frac{1}{1-\lambda}$ and $g(\lambda) := 1-\lambda$. It is easy to show that $\sigma(x,T) = \{1,\frac{1}{2}\}$ and $\sigma((I-T)x,T) = \{\frac{1}{2}\}$, hence $x \notin D(f[T])$ and $(I-T)x \in D(f[T])$. Using the local functional calculus $f[T](I-T)x = (0,1,0,\dots)$ and $(fg)[T]x = x$.

Note that $x \notin D(f[T])$. So we cannot define $g[T]f[T]x$.

McGuire proved in [7] the equality (6) in the case X is a complex separable Hilbert space and T has empty point spectrum.

4. Stability under the action of polynomials

Our first result provides a characterization for the equality $\sigma(p(T)x,T) = \sigma(x,T)$ when p is a polynomial.

Theorem 1. *Assume $T \in L(X)$ satisfies the SVEP. Let $x \in X$ and let*

$$p(\lambda) = (\alpha_1-\lambda)^{n_1}\dots(\alpha_p-\lambda)^{n_p}$$

be a polynomial with $\alpha_i \neq \alpha_j$ for $i \neq j$. We have

$$\sigma(p(T)x,T) = \sigma(x,T)$$

if and only if there is no $i \in \{1,\dots,p\}$ so that α_i is a pole of $\widehat{x}_T$ of order $\leq n_i$.

Consequently, $\sigma(p(T)x,T) = \sigma(x,T)$ if no α_i is an isolated point of $\sigma(x,T)$.

Proof. Assume first $p(\lambda) = (\alpha-\lambda)^n$. We shall prove that

$$\sigma((\alpha-T)^n x,T) \neq \sigma(x,T) \tag{7}$$

if and only if α is a pole of $\widehat{x}_T$ of order $\leq n$.

Taking into account

$$\sigma((\alpha - T)x, T) \subset \sigma(x, T) \subset \sigma((\alpha - T)x, T) \cup \{\alpha\},$$

(7) is equivalent to $\alpha \in \sigma(x, T) \cap \rho((\alpha - T)^n x, T)$. We have that $(\alpha - T)^n \widehat{x}_T$ is the local resolvent of T at $(\alpha - T)^n x$ in a punctured neighborhood of α. Hence $\alpha \in \rho((\alpha - T)^n x, T))$ if and only if the function $(\alpha - T)^n \widehat{x}_T$ is continuous at α.

Considering the binomial expansion

$$(\alpha - T)^n = (\alpha - \mu + \mu - T)^n = \sum_{r=0}^{n} \binom{n}{r} (\alpha - \mu)^r (\mu - T)^{n-r},$$

and denoting

$$A(\mu, \alpha) := \sum_{r=1}^{n-1} \binom{n}{r} (\alpha - \mu)^r (\mu - T)^{n-r},$$

we obtain

$$\begin{aligned} \lim_{\mu \to \alpha} (\alpha - T)^n \widehat{x}_T(\mu) &= \lim_{\mu \to \alpha} \left[(\alpha - \mu)^n \widehat{x}_T(\mu) + A(\mu, \alpha) \widehat{x}_T(\mu) + (\mu - T)^n \widehat{x}_T(\mu)\right] \\ &= \lim_{\mu \to \alpha} (\alpha - \mu)^n \widehat{x}_T(\mu) + (\alpha - T)^{n-1} x. \end{aligned}$$

Taking into account this equality, it is clear that the function $(\alpha - T)^n \widehat{x}_T$ is continuous at α if and only if $\widehat{x}_T$ have a pole at α of order $\leq n$.

In the general case, an iterated application of the previous process yields $\sigma(p(T)x, T) = \sigma(x, T)$. ☐

Corollary 1. *Assume $T \in L(X)$ satisfies the SVEP. Let $p(\lambda)$ be a polynomial having no zeroes in $\sigma_p(T)$. Then*

$$\sigma(p(T)x, T) = \sigma(x, T), \text{ for all } x \in X.$$

Consequently, if $y \in D(p(T)^{-1}) = R(P(T))$, then

$$\sigma(p(T)^{-1} y, T) = \sigma(y, T).$$

Proof. Assume that there exists $x \in X$ such that $\sigma(p(T)x, T) \neq \sigma(x, T)$. By Theorem 1, there is a zero α of the polynomial p with order n which is a pole of $\widehat{x}_T$ of order $k \leq n$, that is $y := \lim_{\mu \to \alpha} (\mu - \alpha)^k \widehat{x}_T(\mu) \neq 0$, but $\lim_{\mu \to \alpha} (\mu - \alpha)^{k+1} \widehat{x}_T(\mu) = 0$. Then

$$\begin{aligned} (\alpha - T)y &= (\alpha - T) \lim_{\mu \to \alpha} (\mu - \alpha)^k \widehat{x}_T(\mu) \\ &= \lim_{\mu \to \alpha} (\alpha - \mu)(\mu - \alpha)^k \widehat{x}_T(\mu) + \lim_{\mu \to \alpha} (\mu - T)(\alpha - \mu)^k \widehat{x}_T(\mu) \\ &= - \lim_{\mu \to \alpha} (\mu - \alpha)^{k+1} x = 0. \end{aligned}$$

Hence $\alpha \in \sigma_p(T)$. Consequently, $\sigma(p(T)x, T) = \sigma(x, T)$ if p has no zeroes in $\sigma_p(T)$. Hence, it is enough to apply the equality $\sigma(p(T)x, T) = \sigma(x, T)$ to $x \in X$ such that $y = p(T)x$. ☐

Corollary 2. *Let $T \in L(X)$ satisfy the SVEP and let $x \in X$. If $p(\lambda)$ is a polynomial having no zeroes in $\sigma_p(T) \cap \sigma(x, T)$, then*

$$\sigma(p(T)x, T) = \sigma(x, T).$$

Proof. If $\sigma(x,T) \neq \sigma(p(T)x,T)$, then there is a zero α of p which is a pole of $\widehat{x}_T$ of order n. Moreover by Corollary 1, $\alpha \in \sigma_p(T)$ hence $\alpha \in \sigma(x,T) \cap \sigma_p(T)$. □

In general, the converse of the above corollary is not true, as the following example shows.

Example 1. Let $B([0,1])$ denote the Banach space of all bounded functions from [0,1] into $\mathbb{C}$, with the supremum norm. For $u \in B([0,1])$ we define $(Tu)(s) = su(s)$ for all $s \in [0,1]$. If $x(t)$ is given by

$$x(t) = \begin{cases} 0, & 0 \leq t \leq \frac{1}{2}, \\ 1, & \frac{1}{2} < t \leq 1, \end{cases}$$

then $\sigma(x,T) = [\frac{1}{2}, 1]$ and $1 \in \sigma(x,T) \cap \sigma_p(T)$. However for $p(\lambda) := 1 - \lambda$, we have

$$(I-T)x(t) = \begin{cases} 0, & 0 \leq t \leq \frac{1}{2}, \\ (1-t), & \frac{1}{2} < t \leq 1, \end{cases}$$

hence $\sigma((I-T)x,T) = [\frac{1}{2}, 1] = \sigma(x,T)$. □

Remark 2. The results of this section have been obtained for $T \in L(X)$ satisfying the SVEP. If T does not satisfy the SVEP, then Corollary 2 and the necessary condition for the equality $\sigma(p(T)x,T) = \sigma(x,T)$ of Theorem 1 are also true.

5. Stability under the action of analytic and meromorphic functions

The following proposition provides a sufficient condition for the equality

$$\sigma(f[T]x,T) = \sigma(x,T),$$

where f is a function of the local functional calculus.

Proposition 5. *Assume $T \in L(X)$ satisfies the SVEP. Let $x \in X$ and let f be an analytic function in a neighborhood of $\sigma(x,T)$. If f has no zeroes in $\sigma(x,T)$, then*

$$\sigma(f[T]x,T) = \sigma(x,T).$$

Proof. By part (ii) of Proposition 3, it is enough to show that $\rho(f[T]x,T) \subset \rho(x,T)$. We denote $h(\lambda) := f(\lambda)^{-1}$ and $y := f[T]x$. By part (ii) of Proposition 3 we have that $h[T]\widehat{y}_T(\lambda)$ is an analytic function on $\rho(y,T)$. Moreover, since $\widehat{x}_T(\mu) \in D(g[T]) \cap D(f[T])$, using Proposition 4 we obtain

$$(\mu - T)h[T]\widehat{y}_T(\mu) = (\mu - T)h[T]f[T]\widehat{x}_T(\mu) = x$$

for all $\mu \in \rho(y,T)$; hence $\rho(y,T) \subset \rho(x,T)$. □

Theorem 2 (Stability of the local spectrum)**.** *Assume that $T \in L(X)$ satisfies the SVEP. Let $x \in X$ and let f be a function analytic in a neighborhood of $\sigma(x,T)$. Let $\alpha_1, \ldots, \alpha_p$ be the zeroes of f in $\sigma(x,T)$ with multiplicities $n_1, \ldots, n_p$, respectively. Then*

$$\sigma(f[T]x,T) = \sigma(x,T)$$

if and only if there is no $i \in \{1, \ldots, p\}$ so that α_i is a pole of $\widehat{x}_T$ of order $\leq n_i$.

Proof. We write $f(\lambda) = p(\lambda)g(\lambda)$, where $g \neq 0$ in $\sigma(x,T)$ and

$$p(\lambda) = \prod_{i=1}^{n} (\alpha_i - \lambda)^{n_i}.$$

By part (ii) of Proposition 1, we obtain that $\sigma(p(T)x,T) \subset \sigma(x,T)$; hence g has no zeroes in $\sigma(p(T)x,T)$. Applying Proposition 5 to the function g we obtain

$$\sigma(f[T]x,T) = \sigma(g[T]p(T)x,T) = \sigma(p(T)x,T),$$

and using Theorem 1 we conclude the proof. □

The result [7, Theorem 1.5] of McGuire may be readily derived from the following corollary.

Corollary 3. *Assume $T \in L(X)$ satisfies the SVEP. Let $x \in X$, and let f be an analytic function in $\sigma(x,T)$. If f has no zeroes in $\sigma_p(T) \cap \sigma(x,T)$, then $\sigma(x,T) = \sigma(f[T]x,T)$.*

Proof. It is clear by Proposition 5 and Corollary 2. □

The following corollary provides characterizations of when an analytic function f satisfies the equality

$$\sigma(f[T]x,T) = \sigma(x,T), \text{ for all } x \in D(f[T]).$$

Corollary 4. *Assume $T \in L(X)$ satisfies the SVEP. If f is an analytic function which is not identically zero on any component of $\triangle(f)$ intersecting $\sigma(T)$, then the following assertions are equivalent:*

(i) *f has no zeroes in $\sigma_p(T) \cap \sigma(x,T)$, for all $x \in D(f[T])$.*
(ii) *$\sigma(f[T]x,T) = \sigma(x,T)$, for all $x \in D(f[T])$.*
(iii) *$f[T]$ is injective.*

Proof. (i) $\Rightarrow$ (ii) It is clear by Corollary 3.

(ii) $\Rightarrow$ (iii) It is enough to note that $\sigma(x,T) = \emptyset$ if and only if $x = 0$.

(iii) $\Rightarrow$ (i) Suppose that f has a zero α belonging to $\sigma_p(T) \cap \sigma(x,T)$, with multiplicity k. We may write $f(\lambda) = g(\lambda)(\alpha - \lambda)^k$.

Then there exists a non-zero vector $y \in X$ such that $(\alpha - T)y = 0$ and $\sigma(y,T) = \{\alpha\}$. Hence $y \in D(f[T])$ and $f[T]y = g[T](\alpha - T)^k y = 0$. □

In the following corollary we give a necessary and sufficient condition for the stability of the local spectrum by the meromorphic calculus.

Notice that the result holds for all $x \in D(f\{T\})$, which in general includes properly $D(f[T])$.

Corollary 5. *Assume $T \in L(X)$ satisfies the SVEP. Let f be a meromorphic function in an open set containing $\sigma(T)$, such that the poles of f are outside the point spectrum of T and f is identically zero in no component of $\triangle(f)$.*

Then $\sigma(f\{T\}x,T) = \sigma(x,T)$ for all $x \in D(f\{T\})$ if and only if f has no zeroes in $\sigma_p(T)$.

Proof. First we assume f has no zeroes in $\sigma_p(T)$. As in the definition of the meromorphic calculus, we write $f\{T\} = g(T)p(T)^{-1}$, where g is an analytic function in $\sigma(T)$ and p is a polynomial such that the zeros of p are not in $\sigma_p(T)$. By Proposition 5 we have

$$\sigma(g(T)p(T)^{-1}x,T) = \sigma(p(T)^{-1}x,T),$$

and by Corollary 1

$$\sigma(p(T)^{-1}x, T) = \sigma(x, T),$$

hence $\sigma(f\{T\}x, T) = \sigma(x, T)$.

For the converse, note that if f has a zero α in $\sigma_p(T)$, then there exists a non-zero vector $x \in X$ such that $(\alpha - T)x = 0$, hence $f\{T\}x = 0$. □

Finally our aim is to provide a property similar to (4) for the operator $f[T]$.

Proposition 6. *Assume $T \in L(X)$ satisfies the SVEP, and let f be an analytic function in $\sigma(x, T)$. Then*

$$\sigma(x, T) \subset \sigma(f[T]x, T) \cup Z_x(f, T),$$

where $Z_x(f, T)$ denotes the set of all zeros of f in $\sigma(x, T)$.

Proof. We can write $f(\mu) = p(\mu)g(\mu)$, with p representing the zeroes of f in $\sigma(x, T)$ and g having no zeroes in $\sigma(x, T)$.

Note that $f[T]x = p(T)g[T]x$. By Proposition 5 we have $\sigma(x, T) = \sigma(g[T]x, T)$, and it is clear that $\sigma(x, T) \subset \sigma(g[T]x, T) \cup Z_x(p, T) = \sigma(p(T)g[T]x, T) \cup Z_x(p, T)$. So the result is proved. □

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Departamento de Análisis Matemático, Universidad de La Laguna, La Laguna, Spain
E-mail address: `tbermudez@ull.es`

Departamento de Matemáticas, Universidad de Cantabria, Santander, Spain

Departamento de Análisis Matemático, Universidad de La Laguna, La Laguna, Spain