PROCEEDINGS OF THE
AMERICAN MATHEMATICAL SOCIETY
Volume 125, Number 2, February 1997, Pages 329–338
S 0002-9939(97)03532-6

ON THE CURVES OF CONTACT ON SURFACES IN A PROJECTIVE SPACE. III

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(Communicated by Wolmer V. Vasconcelos)

Abstract. Suppose a smooth curve C is a set-theoretic complete intersection of two surfaces F and G with the multiplicity of F along C less than or equal to the multiplicity of G along C. One obtains a relation between the degrees of C, F and G, the genus of C, and the multiplicity of F along C in case F has only ordinary singularities. One obtains (in the characteristic zero case) that a nonsingular rational curve of degree 4 in $\mathbf{P}^3$ is not set-theoretically an intersection of 2 surfaces, provided one of them has at most ordinary singularities. The same result holds for a general nonsingular rational curve of degree ≥ 5.

Introduction

In [2] we characterized the smooth curves C which are a set-theoretic complete intersection on a given irreducible surface F in $\mathbf{P}^3$ in case $C \not\subset \mathrm{Sing}F$. In [3] the characterization was made more explicit if $C \cap \mathrm{Sing}F$ consists only of rational double points. Moreover we also characterized the curves of contact on F which are not contained in $\mathrm{Sing}F$ provided F has only ordinary singularities (i.e. those which admit a general projection of a nonsingular surface in $\mathbf{P}^3$ in the characteristic zero case).

The aim of this paper is to study the smooth curves of contact on F in case $C \subset \mathrm{Sing}F$. A useful tool for this study is the symmetric multiple structures. It turns out that the obvious multiple structure defined on C, in case C is a curve of contact on F, is symmetric if $\mathrm{Sing}F$ contains at least one pinch point.

Suppose a smooth curve C is a set-theoretic complete intersection of two surfaces F and G with the multiplicity of F along C less than or equal to the multiplicity of G along C. One obtains a relation between the degrees of C, F and G, the genus of C, and the multiplicity of F along C in case the normal cone to C in the scheme defined by F and G is locally (along C) a complete intersection in the normal bundle to C in $\mathbf{P}^3$. This (rather technical) condition is satisfied if F has only ordinary singularities.

Putting together the results of this paper and those of [3] we obtain (in the characteristic zero case) that a nonsingular rational curve of degree 4 in $\mathbf{P}^3$ is not

Received by the editors July 25, 1994 and, in revised form, December 13, 1994 and June 27, 1995.

1991 *Mathematics Subject Classification.* Primary 14H50.

This research was supported by the funds of the Italian Ministry of Education and Scientific Research (MURST).

set-theoretically an intersection of 2 surfaces, provided one of them has at most ordinary singularities. The same result holds for a general nonsingular rational curve of degree ≥ 5.

Of course the main inspiration for this paper is the problem whether any (connected) curve in $\mathbf{P}^3$ is a set-theoretic complete intersection. The problem is open even in the case of smooth rational curves. It is known ([6]) that a noncomplete intersection curve cannot be a set-theoretic complete intersection on a nonsingular surface. That is the reason why the singular surfaces come into play in this paper.

1. Symmetric multiple structures

In the sequel C will always denote a smooth (connected) curve $\subset \mathbf{P}^3_k$ (k-algebraically closed) and $I \subset \mathcal{O}_{\mathbf{P}^3}$ its ideal sheaf.

Definition. A multiple structure on C is a locally Cohen-Macaulay (lCM) subscheme $\overline{C}$ of $\mathbf{P}^3$ with an ideal sheaf J ($\mathcal{O}_{\mathbf{P}^3}/J$ is locally Cohen-Macaulay) such that $I^{t+1} \subset J \subset I$ for some $t \geq 0$.

For any $i \geq 1$ we define J_i as the minimal ideal sheaf containing $J + I^i$ which defines a lCM subscheme of $\mathbf{P}^3$. So J_i is obtained by removing all the embedded components of $J + I^i$. We have $J_1 = I$ and $J_i = J$ for $i \geq t+1$ where $t+1$ will denote in the sequel the least i such that $J \supset I^i$.

Proposition 1.1 ([1]). *Let $J \subset \mathcal{O}_{\mathbf{P}^3}$ be an ideal sheaf defining a multiple structure on $C \subset \mathbf{P}^3$ and let J_i be the ideal sheaves defined as before for $i \geq 1$. We put moreover $J_0 = \mathcal{O}_{\mathbf{P}^3}$. Then*

1°. *$J_i \supset J_{i+1}$ for $i \geq 0$.*

2°. *J_i/J_{i+1} is a locally free $\mathcal{O}_C$-module.*

3°. *$J_iJ_j \subset J_{i+j}$ and the induced map $J_i/J_{i+1} \otimes J_j/J_{j+1} \to J_{i+j}/J_{i+j+1}$ is generically surjective.*

In the sequel we shall put $E_i = J_i/J_{i+1}$. In particular $E_0 = \mathcal{O}_{\mathbf{P}^3}/I = \mathcal{O}_C$.

Proposition 1.2. *Let $\overline{C}$ be a multiple structure on C. Then*

$$\deg \overline{C} = \left(\sum_{i=0}^{t} \operatorname{rank} E_i\right) \deg C.$$

Proof. Let us consider the exact sequence

$$0 \to E_i(n) \to \mathcal{O}_{\mathbf{P}^3}/J_{i+1}(n) \to \mathcal{O}_{\mathbf{P}^3}/J_i(n) \to 0.$$

By Riemann-Roch and additivity of the Euler-Poincaré characteristic

$$\begin{aligned}\deg E_i + (\operatorname{rank} E_i) \deg C n + \operatorname{rank} E_i(1 - p(C)) + (\deg C_i)n + 1 - p(C_i) \\ = (\deg C_{i+1})n + 1 - p(C_{i+1})\end{aligned}$$

where C_i is a (lCM) curve defined by J_i and $p(C_i)$ is its (arithmetic) genus. Comparing the terms which contain n we obtain that $\deg C_{i+1} = \deg C_i + (\operatorname{rank} E_i) \deg C$. An easy induction completes the proof since $\overline{C} = C_{t+1}$.

Let $\operatorname{Gr}_I(\mathcal{O}_{\mathbf{P}^3})$ denote $\bigoplus_{i\geq 0} I^i/I^{i+1}$ ($I^0 = \mathcal{O}_{\mathbf{P}^3}$) and for any $J \subset I$ the sheaf of graded ideals $\bigoplus_{i\geq 0}(J \cap I^i) + I^{i+1}/I^{i+1} \subset \operatorname{Gr}_I(\mathcal{O}_{\mathbf{P}^3})$ will be denoted by J^* (the sheaf of initial forms of J with respect to the I-adic filtration of $\mathcal{O}_{\mathbf{P}^3}$).

Let J define a multiple structure on $C \subset \mathbf{P}^3$. Then $I^i \subset J_i$ for every i, so there is a map $I^i/I^{i+1} \to J_i/J_{i+1} = E_i$ with $(J \cap I^i) + I^{i+1}/I^{i+1}$ contained in its kernel. So we have an induced map $\varphi\colon \mathrm{Gr}_I(\mathcal{O}_{\mathbf{P}^3})/J^* \to \bigoplus_{0\le i\le t} E_i$.

Proposition 1.3. *Let $x \in C$. Then the following conditions are equivalent:*

1°. *$\varphi_x\colon (\mathrm{Gr}_I(\mathcal{O}_{\mathbf{P}^3})/J^*)_x \to (\bigoplus_{0\le i\le t} E_i)_x$ is an isomorphism.*
2°. *$(\mathrm{Gr}_I(\mathcal{O}_{\mathbf{P}^3})/J^*)_x$ is a (finitely generated) free $\mathcal{O}_{C,x}$-module.*
3°. *$(J_i)_x = (J + I^i)_x$ for $0 \le i \le t$.*
4°. *$(\mathrm{Gr}_I(\mathcal{O}_{\mathbf{P}^3})/J^*)_x$ is a Cohen-Macaulay (CM) local ring.*

Proof. The implication $1° \to 2°$ is obvious. If 2° holds, then $(J + I^i/J + I^{i+1})_x \approx (I^i/(J \cap I^i) + I^{i+1})_x$ is a free $\mathcal{O}_{C,x}$-module for $0 \le i \le t$. We want to prove that $\mathcal{O}_{\mathbf{P}^3,x}/(J + I^i)_x$ is CM for $1 \le i \le t$. We induce on i. If $i = 1$ this is true since $J \subset I$. It is enough to show that $\mathfrak{m}_x$—the maximal ideal of $\mathcal{O}_{\mathbf{P}^3,x}$—is not associated to $(J + I^{i+1})_x$ since $\dim \mathcal{O}_{\mathbf{P}^3,x}/(J + I^{i+1})_x = 1$. Suppose $\mathfrak{m}_x a \in (J + I^{i+1})_x$ for some $a \in \mathcal{O}_{\mathbf{P}^3,x}$. Then $a \in (J + I^i)_x$ since $(J + I^{i+1})_x \subset (J + I^i)_x$ and $\mathcal{O}_{\mathbf{P}^3,x}/(J + I^i)_x$ is CM by the inductive hypothesis. It follows that $a \in (J + I^{i+1})_x$ since $(J + I^i/J + I^{i+1})_x$ is a free $\mathcal{O}_{C,x}$-module. This proves the implication $2° \to 3°$. The implication $3° \to 1°$ also holds since

$$((\mathrm{Gr}_I(\mathcal{O}_{\mathbf{P}^3})/J^*)_i)_x = (I^i/(J \cap I^i) + I^{i+1})_x \approx (J + I^i/J + I^{i+1})_x = (E_i)_x$$

for $0 \le i \le t$. Finally the conditions 2° and 4° are equivalent since $(\mathrm{Gr}_I(\mathcal{O}_{\mathbf{P}^3})/J^*)_x$ is a finite extension of $\mathcal{O}_{\mathbf{P}^3,x}/I_x$ which is regular.

Remark. The conditions above hold if and only if they hold over the completion of $\mathcal{O}_{C,x}$. Moreover there exists a nonempty open $U \subset C$ such that for $x \in U$ they are satisfied.

Definition. Let $\overline{C}$ be a multiple structure on C. Then $\overline{C}$ is called a locally complete intersection (lci) if its ideal sheaf is locally generated by 2 elements.

Definition. Let $\overline{C}$ be a multiple structure on C. Then $\overline{C}$ is called symmetric if $\mathrm{rank} E_i = \mathrm{rank} E_{t-i}$ for $0 \le i \le t$.

Remark. In particular $\mathrm{rank} E_t = 1$.

Proposition 1.4. *Let $\overline{C}$ be a symmetric multiple structure on C. Then $\overline{C}$ is a lci if and ony if the pairings $E_i \otimes E_{t-i} \to E_t$ (considered in Proposition 1.1) are nonsingular for $0 \le i \le t$.*

The proof of Proposition 1.4 is the same as the proof of the corresponding statement in case $\mathrm{rank} I/J_2 = 1$ in [4].

Proposition 1.5. *Let $\overline{C}$ be a lci multiple structure on C. Then the following conditions are equivalent:*

1°. *$\overline{C}$ is symmetric.*
2°. *$J^* \subset \mathrm{Gr}_I(\mathcal{O}_{\mathbf{P}^3})$ is generically a complete intersection.*
3°. *There exists $x \in C$ such that $(J^*)_x \subset \mathrm{Gr}_{I_x}(\mathcal{O}_{\mathbf{P}^3,x})$ is a complete intersection (i.e. $(J^*)_x$ is generated by 2 homogeneous elements).*

Proof. Generically $J_i = J + I^i$ for $0 \le i \le t$ since J_i is obtained by removing all the embedded components of $J + I^i$. Therefore over an open set $U \subset C$

$$\begin{aligned}\mathrm{Gr}_I(\mathcal{O}_{\mathbf{P}^3})/J^* &= \bigoplus_{0\le i\le t} I^i/(J\cap I^i) + I^{i+1} \approx \bigoplus_{0\le i\le t} (J + I^i/J + I^{i+1}) \\ &= \bigoplus_{0\le i\le t} J_i/J_{i+1} = \bigoplus_{0\le i\le t} E_i.\end{aligned}$$

For every $x \in U$, the ideal $(J^*)_x$ is a (ht 2) perfect ideal of $\mathrm{Gr}_{I_x}(\mathcal{O}_{\mathbf{P}^3,x})$ since $\mathrm{Gr}_{I_x}(\mathcal{O}_{\mathbf{P}^3,x})/(J^*)_x$ is a finite free extension of $\mathcal{O}_{\mathbf{P}^3,x}/I_x$ which is a discrete valuation ring. Suppose now that $\overline{C}$ is symmetric. It follows from the local version of Proposition 1.4 that the canonical module of $\mathrm{Gr}_{I_x}(\mathcal{O}_{\mathbf{P}^3,x})/(J^*)_x$ is free of rank 1 if $x \in U$. By Serre's Lemma $(J^*)_x$ is a homomorphic image of a rank 2 projective $\mathrm{Gr}_{I_x}(\mathcal{O}_{\mathbf{P}^3,x})$-module. $\mathrm{Gr}_{I_x}(\mathcal{O}_{\mathbf{P}^3,x})$ is a polynomial ring in 2 variables over a (geometric) discrete valuation ring $\mathcal{O}_{\mathbf{P}^3,x}/I_x$. So $(J^*)_x$ is generated by 2 elements since all the projective $\mathrm{Gr}_{I_x}(\mathcal{O}_{\mathbf{P}^3,x})$-modules are free ([5]). It follows from Nakayama's Lemma that two generators of $(J^*)_x$ can be chosen homogeneous since $(J^*)_x$ is a homogeneous ideal of $\mathrm{Gr}_{I_x}(\mathcal{O}_{\mathbf{P}^3,x})$. So the implication $1^\circ \to 2^\circ$ is proved. 2° obviously implies 3°. It follows from the proof of $1^\circ \to 2^\circ$ that $\mathrm{Gr}_I(\mathcal{O}_{\mathbf{P}^3})/J^* \approx \bigoplus_{0\le i\le t} E_i$ over a non-empty open subset $U \subset C$. So, for every $0 \le i \le t$, $\mathrm{rank}(\mathrm{Gr}_I(\mathcal{O}_{\mathbf{P}^3})/J^*)_i = \mathrm{rank} E_i$ where $(\mathrm{Gr}_I(\mathcal{O}_{\mathbf{P}^3})/J^*)_i$ denotes the i-th homogeneous component of $\mathrm{Gr}_I(\mathcal{O}_{\mathbf{P}^3})/J^*$. Let $x \in C$ be such an element that $(J^*)_x$ is a complete intersection. Then

$$\mathrm{rank}(\mathrm{Gr}_{I_x}(\mathcal{O}_{\mathbf{P}^3,x})/(J^*)_x)_i = \mathrm{rank}(\mathrm{Gr}_{I_x}(\mathcal{O}_{\mathbf{P}^3,x})/(J^*)_x)_{t-i}$$

since the Hilbert function of a homogeneous, finite ht 2 complete intersection is symmetric. It follows that $\mathrm{rank} E_i = \mathrm{rank} E_{t-i}$ and $\overline{C}$ is symmetric. This proves that $3^\circ \to 1^\circ$.

The proof of the implication $1^\circ \to 2^\circ$ shows that, for $x \in C$, $(J^*)_x$ is a complete intersection if $\varphi_x\colon (\mathrm{Gr}_I(\mathcal{O}_{\mathbf{P}^3})/J^*)_x \to (\bigoplus_{0\le i\le t} E_i)_x$ is an isomorphism. So we obtain the following

Proposition 1.6. *Let $\overline{C}$ be a lci symmetric multiple structure on C and let $x \in C$. If $\varphi_x\colon (\mathrm{Gr}_I(\mathcal{O}_{\mathbf{P}^3})/J^*)_x \to (\bigoplus_{0\le i\le t} E_i)_x$ is an isomorphism, then $(J^*)_x \subset \mathrm{Gr}_{I_x}(\mathcal{O}_{\mathbf{P}^3,x})$ is a complete intersection.*

2. Easy commutative algebra

In the sequel I will denote an ideal of a local regular ring R with $\dim R = 3$. Let $f \in R$. We denote by $\deg f$ the largest s such that $f \in I^s$ (the degree of f with respect to the I-adic filtration of R). We put f^*= the image of f in $I^{\deg f}/I^{\deg f+1} \subset \mathrm{Gr}_I(R) = \bigoplus_{i\ge 0} I^i/I^{i+1}$ (the initial form of f with respect to the I-adic filtration of R).

Lemma 2.1 ([7])**.** *Suppose $J = (f,g) \subset I \subset R$. If f^* and g^* form a regular sequence in $\mathrm{Gr}_I(R)$, then $J^* = (f^*, g^*)$ where $J^* = \bigoplus_{i\ge 0}(J\cap I^i) + I^{i+1}/I^{i+1} \subset \mathrm{Gr}_I(R)$.*

Proposition 2.2. *Let $I = (x,y) \subset R$ where x and y are the regular parameters of R and R is complete. Suppose that $J = (f,g) \subset I$ where f and g form a regular sequence. If $f^* \in \mathrm{Gr}_I(R) = (R/I)[X,Y]$ is irreducible (R/I is a discrete valuation ring), then there exists $h \in R$ such that $J = (f,h)$ and $J^* = (f^*, h^*)$.*

Proof. If the Proposition is not true then it follows from Lemma 2.1 that, for any $h \in R$ such that $J = (f, h)$, f^* is a divisor of h^* in $\mathrm{Gr}_I(R)$. So there exists $r_1 \in I^{b-a}$ such that $g - r_1 f \in I^{b+1}$ where $a = \deg f$ and $b = \deg g$. It also follows that $g - r_1 f - r_2 f \in I^{b+2}$ for some $r_2 \in I^{b-a+1}$ since $J = (f, g - r_1 f)$. In this way we obtain a sequence of elements $r_1, r_2, \ldots, r_i, \ldots$ such that $r_i \in I^{b-a+i}$ and, for every i, $g - (r_1 f + r_2 f + \cdots + r_i f) \in I^{b+i}$. Since R is complete, $\sum r_i \in R$ and $g = (\sum r_i) f$. But this is impossible since f and g form a regular sequence.

Proposition 2.3. *Let $I = (X, Y) \subset k[[X, Y, Z]]$ and suppose $J = (XY, g)$ such that $J \supset I^i$ for some $g \in I$ and $i \geq 1$. Then*

1°. *$J = (XY, \alpha X^k + \beta Y^l)$ with α, β invertible $\in k[[X, Y, Z]]$, $k, l \geq 1$.*
2°. *For all $i \geq 1$, $k[[X, Y, Z]]/(J + I^i)$ is Cohen-Macaulay.*

Proof. Let $J = (XY, g)$ such that $J \supset I^i$ for some $g \in I$ and $i \geq 1$. Then $g \mod Y = \alpha X^k$ with $\alpha \in k[[X, Z]]$ invertible and $k \geq 1$. It follows that $g = \alpha X^k + rY$ for some $r \in k[[X, Y, Z]]$. $g \mod X = \beta Y^l$ with $\beta \in k[[Y, Z]]$ invertible and $l \geq 1$. So we obtain that $(r \mod X)Y = \beta Y^l$ and $r = \beta Y^{l-1} + sX$ for some $s \in k[[X, Y, Z]]$. We infer that $g = \alpha X^k + \beta Y^l + sXY$ and $J = (XY, \alpha X^k + \beta Y^l)$. This proves 1°.

In order to prove 2° because of the symmetry of X and Y, we can suppose that $k \leq l$. We obviously have $J + I = I$. Moreover $J + I^i = (XY, X^i, Y^i)$ if $2 \leq i \leq k$, (XY, X^k, Y^i) if $k + 1 \leq i \leq l$ and $J + I^i = J$ for $i \geq l + 1$. It is easy to see that the ideals $J + I^i$ are determinantal and therefore, for all $i \geq 1$, $k[[X, Y, Z]]/(J + I^i)$ is Cohen-Macaulay.

Remark. The multiple structure defined by J on $\mathrm{Spec} k[[X, Y, Z]]/I$ is symmetric only if $k = l$.

Proposition 2.4. *Let $I = (X, Y) \subset k[[X, Y, Z]]$. Then there does not exist $J = (XYZ, g)$ with $g \in I$ such that $J \supset I^i$ for some $i \geq 1$.*

Proof. It suffices to note that $J \mod Z$ is principal whereas $I \mod Z$ is a height 2 ideal.

3. Multiple structures defined by two surfaces

Suppose $C = \mathrm{supp}(F \cap G)$ where F and G are two surfaces in $\mathbf{P}^3$. In the sequel J will denote the ideal sheaf corresponding to the ideal of the homogeneous coordinate ring of $\mathbf{P}^3$ which is generated by the equations of F and G. Obviously J defines a multiple structure on C.

Proposition 3.1. *For the multiple structure $\overline{C}$ defined above $E_t \approx \omega_C(4 - m - n)$ where $m = \deg F$, $n = \deg G$ and ω_C is a canonical bundle on C.*

Proof. The exact sequence $0 \to E_t \to \mathcal{O}/J_{t+1} \to \mathcal{O}/J_t \to 0$ ($\mathcal{O} = \mathcal{O}_{\mathbf{P}^3}$) induces the map $\omega_{\overline{C}} \approx \underline{\mathrm{Ext}}^2(\mathcal{O}/J_{t+1}, \omega_{\mathbf{P}^3}) \to \underline{\mathrm{Ext}}^2(E_t, \omega_{\mathbf{P}^3})$ which is surjective since $\mathcal{O}/J_t$ is lCM and hence $\underline{\mathrm{Ext}}^3(\mathcal{O}/J_t, \omega_{\mathbf{P}^3}) = 0$. At the generic point of $\overline{C}$, $E_t = J_t/J_{t+1} = J_t/J$ is the highest nonvanishing power of the maximal ideal of the corresponding local ring. J_t/J_{t+1} is generically generated by one element since the local ring of $\overline{C}$ at its generic point is Gorenstein. It follows that $\mathrm{rank} E_t = \mathrm{rank} J_t/J_{t+1} = 1$.

We obtain $\underline{\mathrm{Ext}}^2(E_t, \omega_{\mathbf{P}^3}) \approx \omega_{\overline{C}} \otimes \mathcal{O}_C \approx \mathcal{O}_C(m+n-4)$ since also $\underline{\mathrm{Ext}}^2(E_t, \omega_{\mathbf{P}^3})$ is a rank one locally free sheaf on C. We further obtain

$$\begin{aligned} E_t &\approx \underline{\mathrm{Ext}}^2(\underline{\mathrm{Ext}}^2(E_t, \omega_{\mathbf{P}^3}), \omega_{\mathbf{P}^3}) \approx \underline{\mathrm{Ext}}^2(\mathcal{O}_C(m+n-4), \omega_{\mathbf{P}^3}) \\ &\approx \underline{\mathrm{Ext}}^2(\mathcal{O}_C, \omega_{\mathbf{P}^3})(4-m-n) \approx \omega_C(4-m-n) \end{aligned}$$

which was to be proved.

For any $x \in C$ let $f_x \in I_x \subset \mathcal{O}_{\mathbf{P}^3,x}$ be the element corresponding to the equation of F. We denote by $\deg f_x$ and f_x^* respectively the degree and the initial form of f_x with respect to the $\widehat{I}_x$-adic filtration of $\widehat{\mathcal{O}}_{\mathbf{P}^3,x}$. Note that $\deg f_x =$ degree of f_x with respect to the I_x-adic filtation of $\mathcal{O}_{\mathbf{P}^3,x}$. In the same way we define $\deg g_x$ and g_x^* where g_x is the element of I_x corresponding to G.

Let $F \subset \mathbf{P}^3$ be a surface containing C. Then $F \in H^0(I^k(m))$ where $m = \deg F$ and $k \geq 1$ (note a slight abuse of the notation). So F induces a section of $I^k/I^{k+1}(m)$. Note that there exists a unique k such that the induced section of $I^k/I^{k+1}(m)$ is nonzero.

Theorem 3.2. *Suppose $C = \mathrm{supp}(F \cap G)$ with $\deg F = m$ and $\deg G = n$ and suppose that, for every $x \in C$, $J_x^* \subset \mathrm{Gr}_{I_x}(\mathcal{O}_{\mathbf{P}^3,x})$ is a complete intersection. If F defines a nonzero section of $I^k/I^{k+1}(m)$ and G defines a nonzero section of $I^l/I^{l+1}(n))$ with $k \leq l$, then*

1°. *$mn = k(t-k+2)d$ where $d = \deg C$.*
2°. *$\omega^{\otimes k(t-k+2)} \approx \mathcal{O}_C(kn + (t-k+2)(m-4k))$ where ω is a canonical bundle of C.*

In particular $k(t-k+2)(2g-2) = d[kn + (t-k+2)(m-4k)]$ where g denotes the genus of C.

Proof. Let $x \in C$. Then $J_x^* = (h_1, h_2)$ with $k = \deg h_1 \leq \deg h_2$. Moreover the Hilbert function of $\bigoplus_{0\leq i\leq t} E_i$ is equal to the Hilbert function of

$$\mathrm{Gr}_{I_x}(\mathcal{O}_{\mathbf{P}^3,x})/J_x^* = (\mathcal{O}_{\mathbf{P}^3,x}/I_x)[X,Y]/(h_1h_2)$$

since $\varphi\colon \mathrm{Gr}_I(\mathcal{O}_{\mathbf{P}^3})/J^* \to \bigoplus_{0\leq i\leq t} E_i$ is an isomorphism by Proposition 1.3.

$$\mathrm{rank} E_i = \begin{cases} i+1, & 0 \leq i \leq k-1, \\ k, & k \leq i \leq \deg h_2 - 1, \\ k + \deg h_2 - i - 1, & \deg h_2 \leq i \leq t. \end{cases}$$

It follows that $\deg h_2 = t-k+2$ since $\mathrm{rank} E_t = 1$. An easy calculation shows that $\sum_{i=0}^t \mathrm{rank} E_i = k(t-k+2)$. To prove 1° it suffices to apply Proposition 1.1 to the multiple structure $\overline{C}$ and note that $\deg \overline{C} = mn$ (Bezout).

Suppose first that $k = \deg h_1 < \deg h_2$. Then, for each $x \in C$, J_x^* is generated by f_x^* with $\deg f_x^* = k$ and some element of $\mathrm{Gr}_{I_x}(\mathcal{O}_{\mathbf{P}^3,x})$ of degree $t-k+2 > k$. It follows that J_x^* in degree $t-k+1$ is generated by f_x^*. F induces a monomorphism $\mathcal{O}_C(-m) \to I^k/I^{k+1}$. We obtain that

$$J_{t-k+1}^* \approx \mathcal{O}_C(-m) \otimes I^{t-2k+1}/I^{t-2k+2}$$

and

$$E_{t-k+1} \approx S^{t-k+1}(N)/\mathcal{O}_C(-m) \otimes S^{t-2k+1}(N)$$

where N denotes the conormal bundle I/I^2 and $S^i(N)$ its i-th symmetric power. It follows from Proposition 1.4 and Proposition 3.1 ($\overline{C}$ is obviously symmetric) that

$$E_{k-1} \approx \underline{\mathrm{Hom}}(E_{t-k+1}, \omega(4-m-n)) \approx (E_{t-k+1})^* \otimes \omega(4-m-n).$$

So we get

$$S^{k-1}(N) \approx (S^{t-k+1}(N)/\mathcal{O}_C(-m) \otimes S^{t-2k+1}(N))^* \otimes \omega(4-m-n)$$

since $E_{k-1} \approx S^{k-1}(N)$. Extracting the highest exterior powers we obtain that

$$\begin{aligned}(\omega^{\otimes -1}(-4))^{\otimes k(k-1)/2} \approx \mathcal{O}_C(-m)^{\otimes t-2k+2} \otimes (\omega^{\otimes -1}(-4))^{\otimes (t-2k+1)(t-2k+2)/2} \\ \otimes (\omega^{\otimes -1}(-4))^{\otimes -(t-k+1)(t-k+2)/2} \otimes (\omega(4-m-n))^{\otimes k}\end{aligned}$$

since $\Lambda^2 N \approx \omega^{\otimes -1}(-4)$ and, for any i, $\Lambda^{i+1} S^i(N) \approx (\Lambda^2 N)^{\otimes i(i+1)/2}$ (apply the splitting principle). Now an easy (but tedious) calculation concludes the proof.

If $k = \deg h_2$, then $t = 2k-2$ and $E_{k-1} \approx \underline{\mathrm{Hom}}(E_{k-1}, \omega(4-m-n))$. $E_{k-1} = S^{k-1}(N)$ and proceeding as above we obtain 2° with $t = 2k-2$.

Extracting the degrees of both sides of 2° we easily obtain that

$$k(t-k+2)(2g-2) = d[kn + (t-k+2)(m-4k)].$$

Proposition 3.3. *Let J be the ideal sheaf of the multiple structure on $C = \mathrm{supp}(F \cap G)$ which was defined above. Suppose that $\varphi\colon \mathrm{Gr}_I(\mathcal{O}_{\mathbf{P}^3})/J^* \to \bigoplus_{0 \le i \le t} E_i$ is an isomorphism over a (nonempty) open set $U \subsetneq C$. If for all $x \in C - U$ either $f_x^* \in \mathrm{Gr}_{\widehat{I}_x}(\widehat{\mathcal{O}}_{\mathbf{P}^3,x})$ or $g_x^* \in \mathrm{Gr}_{\widehat{I}_x}(\widehat{\mathcal{O}}_{\mathbf{P}^3,x})$ is irreducible, then, for every $x \in C$, $J_x^* \subset \mathrm{Gr}_{I_x}(\mathcal{O}_{\mathbf{P}^3,x})$ is a complete intersection.*

Proof. By Proposition 2.2 the extension of J_x^* to $\mathrm{Gr}_{\widehat{I}_x}(\widehat{\mathcal{O}}_{\mathbf{P}^3,x})$ is a complete intersection if $x \in C - U$. It follows that also $J_x^* \subset \mathrm{Gr}_{I_x}(\mathcal{O}_{\mathbf{P}^3,x})$ is a complete intersection. By Proposition 1.5 the multiple structure $\overline{C}$ is symmetric. Applying Proposition 1.6 we obtain that $J_x^* \subset \mathrm{Gr}_{I_x}(\mathcal{O}_{\mathbf{P}^3,x})$ is a complete intersection if $x \in U$.

Corollary 3.4. *Suppose $C = \mathrm{supp}(F \cap G)$ with $\deg F = m$ and $\deg G = n$ and let the ideal sheaf J of the multiple structure $\overline{C}$ satisfy the hypotheses of Proposition 3.3, i.e. there exists a nonempty open set $U \subsetneq C$ such that $\varphi_x\colon (\mathrm{Gr}_I(\mathcal{O}_{\mathbf{P}^3})/J^*)_x \to (\bigoplus_{0 \le i \le t} E_i)_x$ is an isomorphism for $x \in U$ and for all $x \in C - U$ either $f_x^* \in \mathrm{Gr}_{\widehat{I}_x}(\widehat{\mathcal{O}}_{\mathbf{P}^3,x})$ is irreducible or $g_x^* \in \mathrm{Gr}_{\widehat{I}_x}(\widehat{\mathcal{O}}_{\mathbf{P}^3,x})$ is irreducible. If F defines a nonzero section of $I^k/I^{k+1}(m)$ and G defines a nonzero section of $I^l/I^{l+1}(n)$) with $k \le l$, then*

1°. *$mn = k(t-k+2)d$ where $d = \deg C$.*
2°. *$\omega^{\otimes k(t-k+2)} \approx \mathcal{O}_C(kn + (t-k+2)(m-4k))$ where ω is a canonical bundle of C.*

In particular $k(t-k+2)(2g-2) = d[kn + (t-k+2)(m-4k)]$ where g denotes the genus of C.

Remark. In view of the Remark following Proposition 1.3 the condition which concerns the points of $C - U$ is the only essential hypothesis.

4. Ordinary singularities

Recall that a surface $F \subset \mathbf{P}^3_k$ ($\mathrm{ch}k = 0$) admits ordinary singularities if $\mathrm{Sing}F$ is a curve (possibly reducible) and, for $x \in \mathrm{Sing}F$, $\widehat{\mathcal{O}}_{F,x}$ is one of the following:

1. For almost all $x \in \mathrm{Sing}F$, $\widehat{\mathcal{O}}_{F,x} \approx k[[X,Y,Z]]/(XY)$ is an ordinary double point.
2. $\widehat{\mathcal{O}}_{F,x} \approx k[[X,Y,Z]]/(XYZ)$ is an ordinary triple point.
3. $\widehat{\mathcal{O}}_{F,x} \approx k[[X,Y,Z]]/(X^2 - Y^2Z)$ is a pinch point.

It is well known that if $\mathrm{ch}k = 0$, then a generic projection of any (projective) smooth surface into $\mathbf{P}^3_k$ admits only ordinary singularities.

Theorem 4.1. *Let C be a (smooth) curve contained in the singular locus of a surface $F \subset \mathbf{P}^3$ which along C admits only ordinary singularities and among them at least one pinch point. If there exists a surface $G \subset \mathbf{P}^3$ such that $C = \mathrm{supp}(F \cap G)$, then*

$mn = 2td$ *and*

$\omega^{\otimes 2t} \approx \mathcal{O}_C(2n + tm - 8t)$ *if G is singular along (whole) C or*

$mn = (t+1)d$ *and*

$\omega^{\otimes (t+1)} \approx \mathcal{O}_C(m + (t+1)(n-4))$ *otherwise*

where as before ω is a canonical bundle of $C, m = \deg F$, $n = \deg G$, $d = \deg C$ and t is the least i such that $J \supset I^{i+1}$ (I is the ideal sheaf of C and J is the ideal sheaf corresponding to the ideal generated by F and G).

Proof. Let U denote the (open) set of C which consists of ordinary double points of F. It follows from Proposition 1.3 and Proposition 2.3 that $\varphi_x \colon (\mathrm{Gr}_I(\mathcal{O}_{\mathbf{P}^3})/J^*)_x \to (\bigoplus_{0 \le i \le t} E_i)_x$ is an isomorphism for $x \in U$. Moreover it follows from Proposition 2.4 that F has no ordinary triple points along C. So if $x \in C - U$, then

$$f_x^* = X^2 - Y^2 Z \in \mathrm{Gr}_{\widehat{I}_x}(\widehat{\mathcal{O}}_{\mathbf{P}^3,x}) = k[[Z]][X,Y]$$

is irreducible. The application of Theorem 3.2 with $k = 2$ and $k = 1$ respectively concludes the proof. (Note that in case $k = 1$ the roles of m and n are interchanged.)

Proposition 4.2. *Let C be a (smooth) curve contained in the singular locus of a surface $F \subset \mathbf{P}^3$. If F admits along C only ordinary double points, then $\omega^{\otimes 2} \approx \mathcal{O}_C(2m - 8)$ where $m = \deg F$.*

Remark. Note that we do not make any $C = \mathrm{supp}(F \cap G)$ assumption!

Proof. Recall first that for any locally free sheaf P there is a map $\varphi \colon S^2(P^*) \to (S^2 P)^*$ defined locally by $\varphi(f_1 \otimes f_2)(x \otimes y) = f_1(x) f_2(y) + f_1(y) f_2(x)$ with $f_1, f_2 \in P^*$ and $x, y \in P$.

The surface F defines the map $\mathcal{O}_C(-m) \to I^2/I^3 = S^2(I/I^2)$. Composing the dual map $S^2(I/I^2)^* \to \mathcal{O}_C(m)$ with φ we obtain $\alpha \colon S^2((I/I^2)^*) \to \mathcal{O}_C(m)$. We claim that the induced map $\alpha' \colon (I/I^2)^* \to \underline{\mathrm{Hom}}((I/I^2)^*, \mathcal{O}_C(m))$ is an isomorphism. It suffices to check this at the completion of the local ring at each point of C. So we can assume that $I = (x,y)$ and $\mathcal{O}_C(-m)$ is freely generated by one element e (say). Moreover the map $\mathcal{O}_C(-m) \to I^2/I^3$ associates xy to e. (Note a slight abuse of notation). Let (x^*, y^*) be the dual basis of the basis (x,y) of I/I^2. One checks easily that $\alpha((x^*)^2) = 0$, $\alpha(x^* y^*) = e$ and $\alpha((y^*)^2) = 0$. It follows that α' is an isomorphism. So we obtain that

$$(I/I^2)^* \approx \underline{\mathrm{Hom}}((I/I^2)^*, \mathcal{O}_C(m)) \approx (I/I^2) \otimes \mathcal{O}_C(m).$$

Hence

$$(\Lambda^2(I/I^2))^* \approx \Lambda^2(I/I^2) \otimes \mathcal{O}_C(2m)$$

and

$$(\omega^{\otimes -1}(-4))^* \approx \omega^{\otimes -1}(-4) \otimes \mathcal{O}_C(2m).$$

This implies that $\omega^{\otimes 2} \approx \mathcal{O}_C(2m-8)$ which was to be proved.

Corollary 4.3. *Let C be a smooth rational curve of degree $d \geq 3$ contained in the singular locus of a surface $F \subset \mathbf{P}^3$. Then C is not a set theoretic intersection on F if F has along C ordinary singularities.*

Proof. Suppose F has along C only ordinary double points. Then by Proposition 4.2 $\omega^{\otimes 2} \approx \mathcal{O}_C(2m-8)$. Extracting degrees we obtain $-4 = d(2m-8)$. This is not possible if $d \geq 3$, so F admits along C at least one pinch point and we can apply Theorem 4.1. If the singular locus of G contains C, then $mn = 2td$ and extracting degrees we obtain $-4t = d(2n + mt - 8t)$. Putting $n = 2td/m$ in the last equation we get (after simplifications) the following quadratic equation with respect to m:

$$dtm^2 + 4t(1-2d)m + 4td^2 = 0.$$

So the discriminant $D = 16t^2(1-2d)^2 - 16t^2d^3 \geq 0$. We infer that $(1-2d)^2 - d^3 \geq 0$. It follows that $d^2 - 3d + 1 \leq 0$ since $(1-2d)^2 - d^3 = -(d-1)(d^2-3d+1)$ and $d \geq 3$. If $d^2 - 3d + 1 \leq 0$ holds, then $d \leq (3+\sqrt{5})/2 < 3$. So if C is set-theretically the intersection of F and G, then C is not contained in the singular locus of G. So by Theorem 4.1 $\omega^{\otimes(t+1)} \approx \mathcal{O}_C(m + (t+1)(n-4))$. Taking degrees we obtain $-2(t+1) = d[m + (t+1)(n-4)]$. It follows that $n < 4$ since $-2(t+1) < 0$, $d, m, t+1 > 0$. The cases $n = 1$ and $n = 2$ are not possible since C is not a plane curve and C is not a set-theoretic complete intersection on a quadric. If $n = 3$ we obtain $-2(t+1) = dm - d(t+1)$. But by Theorem 4.1 $d(t+1) = 3m$. So $-2(t+1) = (d-3)m$. This equality cannot hold since $d \geq 3$. It follows that C is not a set theoretic complete intersection on F.

Corollary 4.4. *Let C be a smooth rational curve on a surface $F \subset \mathbf{P}^3_k$ which admits only ordinary singularities. If $\deg C = 4$ or $\deg C \geq 5$ and C is general, then C is not a set theoretic complete intersection on F.*

Proof. By [3] C is not a set theoretic complete intersection on F if $C \not\subset \mathrm{Sing}F$ (the assumption that F is irreducible is not used in the proof). If $C \subset \mathrm{Sing}F$ we apply Corollary 4.3.

Remark. Generality of C means that its normal bundle in $\mathbf{P}^3$ is semistable.

Acknowledgment

This paper owes a great deal to the splendid work of the referee whose criticism contributed enormously to its present form.

References

[1] C. Banica, O. Forster, *Multiplicity structures on space curves*, The Lefschetz centennial conference, part I, Contemp. Math., vol. 58, Amer. Math. Soc., Providence, RI, 1986, pp. 47–64. MR **88C:**32018

[2] M. Boratynski, *On the curves of contact on surfaces in a projective space*, Algebraic K-theory, Commutative Algebra and Algebraic Geometry, Contemp. Math., vol. 126, Amer. Math. Soc., Providence, RI, 1992, 1–8. MR **92m:**14033

[3] M. Boratynski, *On the curves of contact on surfaces in a projective space* II, Rend. Sem. Mat. Univers. Politecn. Torino, Vol. 48. 4 (1990), 439–455. MR **94j:**14031

[4] M. Boratynki, *Locally complete intersection multiple structures on smooth algebraic curves*, Proc. Amer. Math. Soc. **115**, (1992), 877–879. MR **92J:**14039

[5] H. Lindel, *On projective modules over polynomial rings over regular rings*, Algebraic K-theory, Lecture Notes in Math; vol. 966, Springer-Verlag, Berlin, Heidelberg, New York, 1982, pp. 169–179. MR **84d:**13009

[6] L. Robbiano, *A problem of complete intersections*, Nagoya Math. J. **52**, (1973) 129–132. MR **48:**11132

[7] P. Valabrega, G. Valla, *Form rings and regular sequences*, Nagoya Math. J. Vol. 72 (1978), 93–101. MR **80d:**14010

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