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ON THE CURVATURE OF CERTAIN EXTENSIONS OF H-TYPE GROUPS

ISABEL DOTTI

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Dedicated to the memory of Franco Tricerri

ABSTRACT. We show that a one-dimensional solvable extension of an H-type group is symmetric if and only if it has negative curvature.

H-type groups are 2-step nilpotent Lie groups which are natural generalizations of the Iwasawa N-groups associated to semisimple Lie groups of real rank one. They were introduced by A. Kaplan and, together with certain natural 1-dimensional solvable extensions, have been studied by various authors in connection with a number of interesting questions in geometry and analysis (see [K], [B], [C], [D], [DR], [G]).

If N is an H-type group, let $S = AN$ be its canonical one-dimensional solvable extension, endowed with the natural left invariant metric (see Section 2). It is well known that this class of solvable groups include rank one symmetric spaces of noncompact type. Furthermore, such S has always nonpositive curvature ([B], [D]). On the other hand, the following result was stated in [B]:

Theorem. *S has negative sectional curvature if and only if S is symmetric.*

F. Tricerri contacted the author about some difficulties found following the argument in the proof of this result. In particular, the plane exhibited on p. 541 of [B] does not have zero curvature as claimed.

The main purpose of this note is to give an independent proof of the above theorem. We shall express the negative curvature property on S by an algebraic condition on the Lie algebra $\mathfrak{n}$ of N (we call it the NC-condition) and will prove that the only H-type algebras satisfying this condition are the Iwasawa $\mathfrak{n}$ algebras. The basic idea in the proof is to use the NC-condition to define a bilinear multiplication without zero divisors on $R + \mathfrak{z}$ ($\mathfrak{z}$ the center of $\mathfrak{n}$) and then use the classification of H-type algebras with center of dimension $0, 1, 3$ or 7.

L. Vanhecke, jointly with J. Berndt and F. Tricerri, have proved the above result in the case when the H-group N has an even-dimensional center (see [BTV], Section 4.2). Their proof uses the explicit computation of the eigenvalues of the curvature operator in S and its relationship with the eigenvalues of the K_Y operator studied by Szabo ([S]). Also, M. Lanzendorf ([L]) has given, by different methods, an alternative proof of the theorem.

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1. H-type algebras satisfying the NC-condition

In this section we define and discuss the NC-condition on H-type algebras. We start by recalling some definitions and basic facts.

Let $\mathfrak{n}$ be a two-step real nilpotent Lie algebra endowed with an inner product $\langle\, ,\rangle$. Assume $\mathfrak{n}$ has an orthogonal decomposition $\mathfrak{n} = \mathfrak{z} \oplus \mathfrak{v}$, where $\mathfrak{z}$ is a subspace of the center of $\mathfrak{n}$ and $[\mathfrak{v}, \mathfrak{v}] \subset \mathfrak{z}$. Define a linear mapping $J : \mathfrak{z} \to \mathrm{End}(\mathfrak{v})$ by

$$\langle J_Z X, Y\rangle = \langle Z, [X, Y]\rangle \tag{1}$$

(note that J_Z is skew-symmetric). Now $\mathfrak{n}$ is said to be an *H-type algebra* if for any $Z_1, Z_2 \in \mathfrak{z}$

$$J_{Z_1} J_{Z_2} + J_{Z_2} J_{Z_1} = -2\langle Z_1, Z_2\rangle I. \tag{2}$$

The corresponding H-type group is the simply connected Lie group N with Lie algebra $\mathfrak{n}$ endowed with the left invariant metric induced by the inner product $\langle\, ,\rangle$ in $\mathfrak{n}$.

It is easily seen that if $\mathfrak{n}$ is of type H and $\mathfrak{z} \neq 0$, then $\mathfrak{z}$ is the center of $\mathfrak{n}$. If $\mathfrak{z} = 0$, then $\mathfrak{n} = \mathfrak{v}$ is abelian.

Given $X \in \mathfrak{v}$, let $J_{\mathfrak{z}} X = \{J_Z X : Z \in \mathfrak{z}\}$. Clearly (1) implies $(J_{\mathfrak{z}} X)^{\perp} = \ker(\mathrm{ad}_X |\mathfrak{v})$, thus we may consider, for every $X \in \mathfrak{v}$, the orthogonal decomposition

$$\mathfrak{v} = J_{\mathfrak{z}} X \oplus \mathbf{R}X \oplus \mathfrak{w}_X, \tag{3}$$

where $\mathfrak{w}_X$ is the orthogonal complement of $\mathbf{R}X$ in $\ker(\mathrm{ad}_X|\mathfrak{v})$. Furthermore (2) gives

$$\mathrm{ad}_X\, J_Z X = ||X||^2 Z. \tag{4}$$

In the rest of the section we introduce the NC-condition and give a characterization of H-type Lie algebras which satisfy it.

An H-type algebra $\mathfrak{n}$ *satisfies the NC-condition* if $[X, J_{Z_1} J_{Z_2} X] \neq 0$ for every nonzero $X \in \mathfrak{v}$ and any linearly independent Z_1, Z_2 in $\mathfrak{z}$. Equivalently, the projection $P_{J_{\mathfrak{z}} X}(J_{Z_1} J_{Z_2} X)$ onto $J_{\mathfrak{z}} X$, with respect to the decomposition given by (3), is nonzero.

Proposition 1.1. *Let $\mathfrak{n}$ be an H-type algebra satisfying the NC-condition. Then* $\dim(\mathfrak{z})$ *is* 0, 1, 3 *or* 7.

Proof. Fix $X \neq 0$ in $\mathfrak{v}$ and set

$$\tau_X(Z_1, Z_2) = [X, J_{Z_1} J_{Z_2} X], \qquad Z_1,\ Z_2 \in \mathfrak{z}.$$

Then (2) implies $\tau_X(Z_1, Z_2) = -\tau_X(Z_2, Z_1)$ and (1) implies $\langle \tau_X(Z_1, Z_2), Z_1\rangle = 0$ (hence $\langle \tau_X(Z_1, Z_2), Z_2\rangle = 0$).

We next show that the above two properties of τ_X force $\mathfrak{z}$ to have dimension $0, 1, 3$ or 7.

We claim that the bilinear multiplication on $\mathbf{R} \times \mathfrak{z}$

$$(t_1, Z_1)(t_2, Z_2) = (t_1 t_2 - \langle Z_1, Z_2\rangle, t_1 Z_2 + t_2 Z_1 + \tau_X(Z_1, Z_2))$$

has no zero divisors. To prove this assertion assume the right-hand side vanishes. Then

$$t_1 t_2 = \langle Z_1, Z_2\rangle, \quad t_1 Z_2 + t_2 Z_1 = 0, \quad \tau_X(Z_1, Z_2) = 0.$$

Since τ_X does not vanish on any pair of linearly independent vectors, one has $Z_2 = rZ_1$. Thus $t_1t_2 = r|Z_1|^2$ and $(t_1r + t_2)Z_1 = 0$. If $t_2 = -t_1r$, then $-t_1{}^2r = |Z_1|^2r$ implies that either $r = 0$ (hence $(t_2, Z_2) = 0$), or $(t_1, Z_1) = 0$. If $Z_1 = 0$, then $Z_2 = 0$ and $t_1t_2 = 0$; hence $(t_1, Z_1) = 0$ or $(t_2, Z_2) = 0$ as claimed. Since a bilinear multiplication on $\mathbf{R}^n$ with nonzero divisors occurs only in dimensions $1, 2, 4$ and 8, the assertion follows.

The H-type algebras with $\dim \mathfrak{z} = 0, 1, 3, 7$ are constructed as follows (see [K]). Let $\mathbf{F} = \mathbf{R},\ \mathbf{C},\ \mathbf{H}$ or $\mathfrak{o}$. Take $\mathfrak{z} = \operatorname{Im}\mathbf{F}$ ($\mathfrak{z} = 0$ if $\mathbf{F} = \mathbf{R}$), $\mathfrak{v} = \mathbf{F}^p \times \mathbf{F}^q$.
Define

$$[X, Y] = \sum_{l=1}^{p} \operatorname{Im} \bar{x}_l y_l + \sum_{l=p+1}^{q} \operatorname{Im} y_l \bar{x}_l,$$

where $X, Y \in \mathfrak{v},\ X = \sum_{l=1}^n x_l E_l,\ Y = \sum_{l=1}^n y_l E_l,\ x_l, y_l \in \mathbf{F}, n = p + q$ and E_l denotes the element of $\mathbf{F}^n$ with 1 in the l-th position and zero elsewhere.

The inner product on $\mathfrak{z} \oplus \mathfrak{v}$ is given by

$$\langle z + X, u + Y\rangle = \operatorname{Re} \bar{z}u + \sum_{l=1}^{n} \operatorname{Re} \bar{x}_l y_l$$

for $z, u \in \mathfrak{z}$ and $X, Y \in \mathfrak{v}$.

Finally, it follows from the above definitions, that if $z \in \mathfrak{z},\ J_z$ is given by

$$J_z \sum_{l=1}^{n} x_l E_l = \sum_{l=1}^{p} x_l\, z\, E_l + \sum_{l=p+1}^{n} z\, x_l\, E_l,$$

and the resulting algebra, $\mathfrak{n}(\mathbf{F}, p, q)$, is an H-type algebra.

Two H-type algebras are said to be isomorphic if there exists an orthogonal Lie algebra isomorphism between them. For example, it is immediate that

$$\mathfrak{n}(\mathbf{F}, p, q) \simeq \mathfrak{n}(\mathbf{F}, p + q, 0) \quad \text{if } \mathbf{F} = \mathbf{R} \text{ or } \mathbf{F} = \mathbf{C}. \tag{5}$$

Theorem 1.1. *The H-type algebras satisfying the NC-condition are* $\mathfrak{n}(\mathbf{F}, p, 0)$ *if* $\mathbf{F} = \mathbf{R},\ \mathbf{C},\ \mathbf{H}$ *and* $p \in \mathbf{N}$ *or* $\mathfrak{n}(\mathfrak{o}, 1, 0)$.

Proof. It is not hard to verify that the listed algebras satisfy the NC-condition.

Assume now $\mathfrak{n}(\mathbf{F}, p, q)$ satisfies the NC-condition. Because of (5) it suffices to look at the cases $\mathbf{F} = \mathbf{H},\ \mathbf{F} = \mathfrak{o}$.

If $pq \neq 0$, we let $X = E_1 + E_{p+1}$. Then

$$[X, J_i J_j X] = 0 \quad \text{if } \mathbf{F} = \mathbf{H}, \quad \text{and} \quad [X, J_{(0,1)} J_{(0,i)} X] = 0 \quad \text{if } \mathbf{F} = \mathfrak{o}.$$

Thus, either $p = 0$ or $q = 0$ and using the fact that $\mathfrak{n}(\mathbf{F}, p, q)$ and $\mathfrak{n}(\mathbf{F}, q, p)$ are isomorphic, we may assume $p \neq 0$. It remains to show that $p = 1$, if $\mathbf{F} = \mathfrak{o}$. Assume $p > 1$ and let $X = E_1 + (0, 1)E_2$. Then one finds that $[X, J_{(i,0)} J_{(j,0)} X] = 0$ and the theorem follows.

2. Curvature of solvable extensions

In this section we will study the curvature on S, a solvable extension of an H-type group, and will show that the curvature being negative on S corresponds to the NC-condition defined in Section 1.

The class of solvable extensions of H-groups which we will consider are constructed as follows.

Let $\mathfrak{n}$ be an H-type algebra with corresponding simply connected Lie group N. If $A = \mathbf{R}^+$ acts on N by the dilations $(z,x) \to (tz, t^{\frac{1}{2}}x)$, we call S the semidirect product AN. Let $\mathfrak{s}$ be the Lie algebra of S. If D is the derivation of $\mathfrak{n}$ given by $D|_{\mathfrak{v}} = \frac{1}{2} I$ and $D|_{\mathfrak{z}} = I$, and if $\mathfrak{a} = \mathbf{R}A$, then $\mathfrak{s}$ is the semi-direct product $\mathfrak{s} = \mathfrak{a} \oplus \mathfrak{n}$ where $\mathfrak{a}$ acts on $\mathfrak{n}$ via $\mathrm{ad}_A|_{\mathfrak{n}} = D$. We endow $\mathfrak{s}$ with the only inner product extending the given one in $\mathfrak{n}$ and such that $|A| = 1,\ \langle A, \mathfrak{n}\rangle = 0$. Finally, we give to S the Riemannian structure obtained by left translating the inner product on $\mathfrak{s}$.

Standard formulas for left invariant metrics yield the following expression, for the sectional curvature $K(P)$ of a plane P. Let P be spanned by $\{aA + b(Z_1 + X_1), Z_2 + X_2\}$ with $a^2 + b^2 = 1$, $\langle Z_1, Z_2\rangle + \langle X_1, X_2\rangle = 0$ and $|Z_i|^2 + |X_i|^2 = 1,\ i = 1, 2$. We then have

$$K(P) = -\frac{a^2}{4} - \frac{3}{4}|aZ_2 + b[X_1, X_2]|^2 - \frac{3}{4}b^2\langle Z_1, Z_2\rangle^2 - \frac{3}{4}b^2T(P), \tag{6}$$

where $T(P) = |Z_1|^2|Z_2|^2 + 2\langle J_{Z_1}X_1, J_{Z_2}X_2\rangle + \frac{1}{3}$. Note that

$$T(P) \geq |Z_1|^2|Z_2|^2 - 2|Z_1||Z_2||X_1||X_2| + \frac{1}{3} \geq \left(\sqrt{3}|Z_1||Z_2| - \frac{1}{\sqrt{3}}\right)^2 \geq 0 \tag{7}$$

(see [B], p. 539).

It follows from (6) and (7) that

Theorem 2.1. (i) *S has nonpositive sectional curvature* ([B], [D]).

(ii) *S has negative curvature if and only if $\mathfrak{n}$ satisfies the NC-condition.* (*Compare* [S], [BTV].)

Proof. The (i) part follows immediately since in (6) the sectional curvature is expressed as a sum of nonpositive terms. To prove (ii), assume there exists a plane P with zero curvature. Then P is spanned by an orthonormal basis $\{Z_1 + X_1, Z_2 + X_2\}$ satisfying (see (6))

$$\langle Z_1, Z_2\rangle = \langle X_1, X_2\rangle = 0\ ,\quad [X_1, X_2] = 0\ ,$$

and

$$0 = T(P) = |Z_1|^2|Z_2|^2 - 2|Z_1||Z_2||X_1||X_2| + \frac{1}{3} + 2r|Z_1||Z_2||X_1||X_2|,\quad 0 \leq r \leq 2.$$

In particular $r = 0$, hence $J_{Z_1}X_1 = cJ_{Z_2}X_2,\ c < 0$ and $|Z_1||Z_2| = \frac{1}{3},\ |X_1||X_2| = \frac{2}{3}$. Finally, using that $|Z_i|^2 + |X_i|^2 = 1,\ i = 1, 2$, one obtains

$$|Z_1| = |Z_2| = \frac{1}{\sqrt{3}},\quad |X_1| = |X_2| = \sqrt{\frac{2}{3}},\quad J_{Z_1}X_1 = -J_{Z_2}X_2.$$

Thus $\mathfrak{n}$ does not satisfy the NC-condition. To prove the converse let $X \neq 0$ and let Z_1, Z_2 be linearly independent in $\mathfrak{z}$ such that $[X, J_{Z_1}J_{Z_2}X] = 0$. We may assume $|Z_1| = |Z_2| = \sqrt{\frac{1}{3}},\ |X| = \sqrt{\frac{2}{3}}$ and $\langle Z_1, Z_2\rangle = 0$. It is easy to verify that $\{Z_1 + X, Z_2 - 3J_{Z_1}J_{Z_2}X\}$ spans a plane with zero curvature and the proposition follows.

The algebraic characterization of the negative curvature condition together with the classification obtained in Theorem 1.1 imply

Theorem. *S has negative sectional curvature if and only if S is symmetric.*

Proof. Assume S is symmetric. Let $X, Y \in \mathfrak{v}$, $\langle X, Y\rangle = 0$ and $[X, Y] = 0$. Then one computes $0 = \nabla_X R(Z, Y, Y) = \frac{1}{4} J_{[X, J_Z Y]} Y$. Thus J_Z preserves $\mathfrak{w}_X$ (see (3)) hence it preserves $J_{\mathfrak{z}} X \oplus \mathbf{R}X$. Since $\langle J_{Z_1} J_{Z_2} X, J_{\mathfrak{z}} X\rangle = 0$ implies that Z_1, Z_2 are linearly dependent, the projection $P_{J_{\mathfrak{z}} X}(J_{Z_1} J_{Z_2} X)$ onto $J_{\mathfrak{z}} X$, with respect to the decomposition given by (3), is nonzero and $\mathfrak{n}$ satisfies the NC-condition. Conversely, if S has negative sectional curvature, $\mathfrak{n}$ satisfies the NC-condition and by Theorem 1.1 $\mathfrak{n}$ is the Lie algebra of an Iwasawa N-group. It is well known that the solvable extensions of these groups give Riemannian symmetric manifolds. Thus, the main theorem follows.

2.1. In [C], Cowling et al. defined and studied the J^2-condition on an H-type algebra. This property (considered also by Heintze in [H]) is closely related to the NC-condition. We recall its definition.

An H-type algebra $\mathfrak{n}$ *satisfies the* J^2*-condition* if for every $X \in \mathfrak{v}$ the subspace $\mathbf{R}X \oplus J_{\mathfrak{z}} X$ is J_Z-invariant, for all $Z \in \mathfrak{z}$. In particular, if $X \in \mathfrak{v}$ and $Z_1, Z_2 \in \mathfrak{z}$ with $\langle Z_1, Z_2\rangle = 0$, there exists $Z_3 \in \mathfrak{z}$ such that $J_{Z_1} J_{Z_2} X = J_{Z_3} X$. It follows that H-type algebras which fulfil the J^2-condition also satisfy the NC-condition. Moreover, the H-type algebras in Theorem 1.1 do have this property. We therefore have

(8) $\mathfrak{n}$ satisfies the NC-condition if and only if $\mathfrak{n}$ satisfies the J^2-condition.

The NC-condition can also be formulated in terms of the operator K_Y, $Y \in \mathfrak{n}$ studied by Szabo in [S]. For every $Y \in \mathfrak{n}$, $Y = Z + X$ and $\langle Z', Z\rangle = 0$ we set $K_Y(Z') = [\overline{X}, J_{\overline{Z}} J_{Z'} X]$, where $\overline{X} = X/|X|$, $\overline{Z} = Z/|Z|$. It is clear that K_Y is skew symmetric and that $\mathfrak{n}$ satisfies the NC-condition if and only if K_Y is an isomorphism, for every $Y = Z + X$, $Z \neq 0, X \neq 0$. Moreover, in Theorem 1.11 of [S], also in Section 4.2 of [BTV], the eigenvalues and the corresponding eigenspaces of the curvature operator in S are computed. It is shown that they depend on the eigenvalues of ${K_Y}^2$. As a consequence one can deduce (see the end of Section 4.2 in [BTV]) that S has negative sectional curvature if and only if 0 is not an eigenvalue of ${K_Y}^2$, for any $Y = Z + X$, $Z \neq 0, X \neq 0$, which is equivalent to (ii) in Theorem 2.1. Furthermore, as observed in [BTV], this characterization implies that when $\mathfrak{z}$ is even dimensional there exist planes of zero curvature.

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FaMAF, Universidad Nacional de Córdoba, 5000 Córdoba, Argentina
E-mail address: idotti@mate.uncor.edu