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EVERY NONREFLEXIVE SUBSPACE OF $L_1[0,1]$ FAILS THE FIXED POINT PROPERTY

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Abstract. The main result of this paper is that every nonreflexive subspace Y of $L_1[0,1]$ fails the fixed point property for closed, bounded, convex subsets C of Y and nonexpansive (or contractive) mappings on C. Combined with a theorem of Maurey we get that for subspaces Y of $L_1[0,1]$, Y is reflexive if and only if Y has the fixed point property. For general Banach spaces the question as to whether reflexivity implies the fixed point property and the converse question are both still open.

Introduction

We introduce the notion of an asymptotically isometric copy of ℓ_1 and use it to show that every nonreflexive subspace of $L_1[0,1]$ fails the fixed point property for nonexpansive mappings, proving the converse of a theorem of Maurey [M]. In particular, the Hardy space H^1 on the unit circle must fail to have the fixed point property, which contrasts with Maurey's result in [M] that H^1 has the weak (and weak-star) fixed point property.

We only deal with the failure of the fixed point property (FPP) in this paper. The failure of the weak FPP for the Banach space $(L_1[0,1], \|\cdot\|_1)$ was discovered by Alspach [A]. This is still (apart from its superspaces) the only Banach space known to fail the weak FPP. On the other hand the ultrapower techniques of Maurey [M] have been extended to prove the weak FPP in many spaces. Examples of such spaces are: $(c_0, \|\cdot\|_\infty)$ ([M]), the Tsirelson space of Figiel and Johnson (Elton et al. [ELOS]) and every Banach space with an unconditional basis, constant $< (\sqrt{33}-3)/2$, ([Lin]).

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0. Preliminaries

Recall that ℓ_1 is the Banach space of all scalar sequences $x = (x_n)_{n=1}^{\infty}$ for which $\|x\|_1 := \sum_{n=1}^{\infty} |x_n| < \infty$. $L_1[0,1]$ is the usual space of Lebesgue integrable functions (where almost everywhere equal functions are identified), with its usual norm.

Let $(X, \|\cdot\|_X)$ be a Banach space. We say that $(X, \|\cdot\|_X)$ has the fixed point property (FPP) if given any non-empty, closed, bounded and convex subset C of X, every nonexpansive mapping $T : C \to C$ has a fixed point. Here T is nonexpansive if $\|Tx - Ty\|_X \leq \|x - y\|_X$ for all $x, y \in C$. Moreover, T is a contraction if $\|Tx - Ty\|_X < \|x - y\|_X$ for every $x, y \in C$ with $x \neq y$. If X is a dual space, isometrically isomorphic to Y^* for some Banach space Y, then $(X, \|\cdot\|_X)$ has the weak-star fixed point property (with respect to Y) if given a non-empty, weak-star compact, convex set C in X, every nonexpansive mapping on C has a fixed point. The weak fixed point property is defined analogously.

1. All nonreflexive subspaces of $L_1[0,1]$ fail the FPP

1.1 Definition. We say that a Banach space $(X, \|\cdot\|_X)$ is *asymptotically isometric to* ℓ_1 if it has a normalized Schauder basis $(x_n)_{n=1}^{\infty}$ such that for some sequence $(\lambda_n)_{n=1}^{\infty}$ in $(0, \infty)$ increasing to 1, we have that

$$(\spadesuit) \qquad \sum_{n=1}^{\infty} \lambda_n \, |t_n| \leq \left\| \sum_{n=1}^{\infty} t_n \, x_n \right\|_X$$

for all $x = \sum_{n=1}^{\infty} t_n \, x_n \in X$.

Note that whenever $(X, \|\cdot\|_X)$ contains a normalized sequence $(x_n)_{n=1}^{\infty}$ satisfying $(\spadesuit)$, then the closed linear span of $(x_n)_{n=1}^{\infty}$ is an asymptotically isometric copy of ℓ_1.

1.2 Theorem. *Let $(Y, \|\cdot\|_Y)$ be a Banach space containing an asymptotically isometric copy of ℓ_1. Then $(Y, \|\cdot\|_Y)$ fails the fixed point property for closed, bounded, convex sets in Y and nonexpansive (or contractive) maps on them.*

Proof. Let $(x_n)_{n=1}^{\infty}$ in Y and $(\lambda_n)_{n=1}^{\infty}$ satisfy $(\spadesuit)$ above. Now fix a sequence $(\mu_n)_{n=1}^{\infty}$ satisfying $\mu_n > \mu_{n+1}$ for all $n \in \mathbf{N}$, with $\mu_n \underset{n}{\to}$ some real number $r > 0$. Each $\mu_{n+1}/\mu_n \in (0,1)$, so that by passing to corresponding subsequences of $(x_n)_{n=1}^{\infty}$ and $(\lambda_n)_{n=1}^{\infty}$ (if necessary), we may ensure that

$$\lambda_n > \frac{\mu_{n+1}}{\mu_n} \quad , \quad \text{for all} \quad n \in \mathbf{N} \; .$$

Now define $e_n := \mu_n \, x_n$, for all $n \in \mathbf{N}$, and let

$$K := \left\{ \sum_{n \in \mathbf{N}} \alpha_n \, e_n : \text{ each } \alpha_n \geq 0 \text{ and } \sum_{n \in \mathbf{N}} \alpha_n = 1 \right\} \quad .$$

Clearly, K is closed and convex in Y. K is bounded since $\mu_n \underset{n}{\to} r \in (0, \infty)$. Define $T : K \to K$ to be the right shift map; i.e.

$$T\left(\sum_{n \in \mathbf{N}} \alpha_n \, e_n \right) := \sum_{n \in \mathbf{N}} \alpha_n \, e_{n+1} \quad .$$

Of course, T is fixed point free on K. Finally, we show that T is contractive on K. Fix $z := \sum_{n\in\mathbf{N}} \alpha_n\, e_n$ and $w := \sum_{n\in\mathbf{N}} \beta_n\, e_n$ in K, with $z \neq w$. Then,

$$\begin{aligned}\|Tz - Tw\|_Y &= \left\|\sum_{n\in\mathbf{N}} (\alpha_n - \beta_n)\ e_{n+1}\right\|_Y \leq \sum_{n\in\mathbf{N}} |\alpha_n - \beta_n|\ \|e_{n+1}\|_Y \\ &= \sum_{n\in\mathbf{N}} |\alpha_n - \beta_n|\ \mu_{n+1} < \sum_{n\in\mathbf{N}} |\alpha_n - \beta_n|\ \lambda_n\ \mu_n \\ &\leq \left\|\sum_{n\in\mathbf{N}} (\alpha_n - \beta_n)\ \mu_n\ x_n\right\|_Y = \|z - w\|_Y \quad .\end{aligned}$$

□

Immediately we have the following corollary.

1.3 Corollary. *Let $(X, \|\cdot\|_X)$ be a Banach space and Y be a subspace of X such that there exists a sequence $(v_n)_{n=1}^\infty$ in Y, a sequence $(u_n)_{n=1}^\infty$ in X and a null sequence $(\gamma_n)_{n=1}^\infty$ in $(0,\infty)$ with the following properties.*

(i) $\left\|\sum_{n=1}^N t_n u_n\right\|_X = \sum_{n=1}^N |t_n|$, *for all scalar sequences $t_1, \ldots, t_N$ and $N \in \mathbf{N}$.*

(ii) $\|u_n - v_n\|_X < \gamma_n$, *for all $n \in \mathbf{N}$.*

Then $(Y, \|\cdot\|_X)$ fails the fixed point property for closed, bounded, convex sets in Y and nonexpansive (or contractive) mappings on them.

Proof. Without loss of generality, each $\gamma_n < 1$ and $(v_n)_{n=1}^\infty$ is normalized. Then $(v_n)_{n=1}^\infty$ spans an asymptotically isometric copy of ℓ_1 in $(Y, \|\cdot\|_X)$, with the λ_n's in inequality (♠) above given by $\lambda_n := 1 - \gamma_n$, for all $n \in \mathbf{N}$. □

1.4 Theorem. *Every nonreflexive subspace Y of $L_1[0,1]$, with its usual norm, fails the fixed point property for closed, bounded, convex sets in Y and nonexpansive (or contractive) mappings on them. In particular, this is true for $Y := H^1(\mathbf{T})$, the usual Hardy space on the unit circle $\mathbf{T}$.*

Proof. By the proof of the Kadec-Pełczynski theorem [KP] (or see [D, Chapter VII]), for $X := L_1[0,1]$ with its usual norm, sequences $(v_n)_{n=1}^\infty$ in Y, $(u_n)_{n=1}^\infty$ in X and $(\gamma_n)_{n=1}^\infty$ in $(0,\infty)$ exist that satisfy the hypotheses of Corollary 1.3 above. □

Combining 1.4 with Maurey's theorem [M] allows us to state the fact below.

1.5 Theorem. *Let Y be a subspace of $L_1[0,1]$ with its usual norm. Then the following are equivalent.*
(i) *Y is reflexive.*
(ii) *Y has the fixed point property.*

2. Notes and remarks

The basic problem that is still open is: "If X is a Banach space isomorphic to ℓ_1, does X fail the FPP?" Our results only provide a partial answer because there do exist Banach spaces X isomorphic to ℓ_1, that contain *no* asymptotically isometric copies of ℓ_1. These are described in the recent paper of Dowling et al. [DJLT]. In contrast, in Dowling et al. [DLT] the authors show that the spaces ℓ_∞ and $\ell_1(\Gamma)$, with Γ uncountable, cannot be equivalently renormed to have the FPP. Indeed, all such renormings contain asymptotically isometric copies of ℓ_1. This leads to the fact that for a broad class of Orlicz spaces with the Orlicz norm, reflexivity is equivalent to the FPP. Moreover, in Dodds et al. [DDDL], it is shown that every nonreflexive subspace of the trace class $\mathcal{C}_1$ or the predual $\mathcal{M}_*$ of a von Neumann algebra $\mathcal{M}$ with a faithful, normal, finite trace τ contains an asymptotically isometric copy of ℓ_1. Further, in Carothers et al. [CDL] the analogous result for nonreflexive subspaces of the Lorentz function space $L_{w,1}(0,\infty)$ is established. Indeed, for subspaces of $\mathcal{C}_1$ and $L_{w,1}(0,\infty)$ with a strictly decreasing weight function w, the analogue of Theorem 1.5 is true (see [DDDL, CDL]). The situation where ℓ_1 is replaced by c_0 is also considered in [DJLT] and [DLT].

The ideas herein were partially inspired by an example of Lim [Lim]. Smyth [S] has extended the approach based on Lim's example to show that the dual of every space $C(\Omega)$, where Ω is an infinite compact Hausdorff space, fails the weak-star fixed point property with an affine contraction. In particular, ℓ_1 fails the weak-star fixed point property with respect to its predual c (the space of all convergent sequences) with a contractive, affine map.

References

[A] D.E. Alspach, *A fixed point free nonexpansive map*, Proc. Amer. Math. Soc. **82** (1981), 423-424. MR **82j:**47070

[CDL] N.L. Carothers, S.J. Dilworth and C.J. Lennard, *On a localization of the UKK property and the fixed point property in $L_{w,1}$*, Interaction Between Functional Analysis, Harmonic Analysis, and Probability (Columbia, MO, 1994), Lecture Notes in Pure and Appl. Math., vol. 175, Dekker, New York, 1966, pp. 111–124.

[D] J. Diestel, *Sequences and Series in Banach Spaces*, Springer-Verlag, New York Inc., 1984. MR **85i:**46020

[DDDL] P.G. Dodds, T.K. Dodds, P.N. Dowling and C.J. Lennard, *Asymptotically isometric copies of ℓ_1 in nonreflexive subspaces of symmetric operator spaces*, in preparation.

[DJLT] P.N. Dowling, W.B. Johnson, C.J. Lennard and B. Turett, *The optimality of James's distortion theorems*, Proc. Amer. Math. Soc. (to appear).

[DLT] P.N. Dowling, C.J. Lennard and B. Turett, *Reflexivity and the fixed point property for nonexpansive maps*, J. Math. Anal. Appl. **200** (1996), 653–652.

[ELOS] J. Elton, P. Lin, E. Odell and S. Szarek, *Remarks on the fixed point problem for nonexpansive maps*, Contemporary Math., vol. 18, A.M.S., Providence, RI, 1983, pp. 87-119. MR **85d:**47059

[KP] M.I. Kadec and A. Pełczynski, *Bases, lacunary sequences and complemented subspaces in the spaces L_p*, Studia Math. **21** (1962), 161–176. MR **27:**2851

[Lim] T.C. Lim, *Asymptotic centers and nonexpansive mappings in conjugate Banach spaces*, Pacific J. Math. **90** (1) (1980), 135-143. MR **82h:**47052

[Lin] Pei-Kee Lin, *Unconditional bases and fixed points of nonexpansive mappings*, Pacific J. Math. **116** (1) (1985), 69-76. MR **86c:**46075

[M] B. Maurey, *Points fixes des contractions de certains faiblement compacts de L^1, Seminaire d'Analyse Fonctionelle, Exposé no.* VIII, École Polytechnique, Centre de Mathématiques (1980-1981). MR **83h:**47041

[S] M. Smyth, *Remarks on the weak star fixed point property in the dual of $C(\Omega)$*, J. Math. Anal. Appl. **195** (1995), 294–306.

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