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BOHR ALMOST PERIODIC MAPS INTO $K(\pi, 1)$ SPACES

SOL SCHWARTZMAN

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ABSTRACT. Let X be a locally finite simplicial complex of finite topological dimension. Assume further that X is a $K(\pi, 1)$ space where π is a group whose only abelian subgroups are infinite cyclic. We prove that a Bohr almost periodic map of the real line into X is uniformly homotopic to a periodic map. As a consequence we show that a Bohr almost periodic geodesic on a compact Riemannian manifold of everywhere negative curvature is necessarily periodic.

If X is a compact metric space, a map f of the real line into X is said to be Bohr almost periodic provided for any $\varepsilon > 0$ the set of all t_0 such that $\rho(f(t+t_0), f(t)) < \varepsilon$ for all t has the property that it intersects every sufficiently large interval. Since the uniformity on a compact Hausdorff space is unique, the question of whether f is Bohr almost periodic depends only on the topology of X and not on the particular metric we use. If X is any metric space and we do not assume compactness, we say that f is Bohr almost periodic provided the closure of the image of f is compact and f, considered as a map into this closure, is Bohr almost periodic.

If X is imbedded as a closed subset of some Euclidean space R^n and $f(t) = (x_1(t), \ldots, x_n(t))$, then f is Bohr almost periodic if and only if each $x_i(t)$ is Bohr almost periodic. A real valued function is Bohr almost periodic if and only if it can be approximated arbitrarily closely by a trigonometric polynomial. We can associate with each real valued Bohr almost periodic function a Fourier series, which determines the function.

In what follows X will always be the underlying space of a locally finite simplicial complex of finite topological dimension. Finite topological dimension is taken to imply that the space is separable and metrizable; for a locally finite simplicial complex it amounts to assuming that there are only countably many simplices and these are of bounded dimension.

In addition we will assume that X is a $K(\pi, 1)$ space, where π is a group whose only abelian subgroups are infinite cyclic. If X were a compact Riemannian manifold whose sectional curvature was everywhere negative it would satisfy all these conditions [2]. In particular a compact surface of negative Euler characteristic satisfies these conditions, since it admits a Riemannian metric of constant negative curvature.

Let f be a Bohr almost periodic map of the real line into such a $K(\pi, 1)$ space X. We are going to prove the following:

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Theorem 1. *There exists a uniformly continuous map F of $R^1 \times [0,1]$ into X such that*
(1) $F(t,0) = f(t)$ *for all* t,
(2) $F(t,1)$ *is periodic of some period* P.

As a consequence of this result we will show that if X is a compact Riemannian manifold of everywhere negative sectional curvature, any Bohr almost periodic geodesic on X is actually periodic.

In case X is a compact surface of negative Euler characteristic this has been known for a long time [1, 3]. As explained in the sources cited, one can intuitively look upon the result in the following way: Suppose a man takes his dog for a walk in the space X and the dog travels an almost periodic path. If the dog is led by a man using a leash of bounded length the man can lead the dog by this leash provided he travels a suitable periodic path.

If X is a $K(\pi,1)$ space of the kind described above, its one point compactification is of finite topological dimension and therefore can be imbedded in S^n for some n. Thus X can be imbedded as a closed set in n-dimensional Euclidean space R^n for some n. We let i denote such an imbedding. Moreover, X is an absolute neighborhood retract. We choose an open set O containing $i(X)$ and let R be a retraction of O onto X.

If X and Y are any two metric spaces and f_1 and f_2 are two continuous maps of X into Y, we will say that f_1 and f_2 are uniformly homotopic provided there exists a uniformly continuous map of $X \times [0,1]$ into Y such that $h(x,0) = f_1(x)$ and $h(x,1) = f_2(x)$. [Assuming ρ is a metric on X we put a metric $\overline{\rho}$ on $X \times [0,1]$ by defining $\overline{\rho}((x_1,t_1),(x_2,t_2))$ to equal $\rho(x_1,x_2) + |t_1 - t_2|$.] Uniform homotopy of maps between metric spaces is an equivalence relation and the set of such equivalence classes of maps forms a category. If X is compact the uniform homotopy equivalence classes of maps of X into Y are just the ordinary homotopy classes of maps. Moreover if $f : X \to Y$ is uniformly continuous and g_1 and g_2 are uniformly homotopic maps of Y into Z, then $g_1 \circ f$ is uniformly homotopic to $g_2 \circ f$.

Now suppose f is a Bohr almost periodic function of the real line into a $K(\pi,1)$ space X of the type described in the introduction. We are going to show first that there is a homomorphism λ_1 of the real line into a compact connected metrizable abelian group M^* and a map p_1 of M^* into X such that $f = p_1 \circ \lambda$. Then we will show that there is a continuous homomorphism λ_2 of M^* onto a toral group M^*/A_{α_0} (where A_{α_0} will be a closed subgroup of M^*) and a continuous function p_2 of M^*/A_{α_0} into X such that p_1 and $p_2 \circ \lambda_2$ are uniformly homotopic. Finally we will get a continuous homomorphism λ_3 of M^*/A_{α_0} into the multiplicative group S^1 of complex numbers of absolute value one and a continuous map p_3 of S_1 into X such that p_2 and $p_3 \circ \lambda_3$ are uniformly homotopic. It will follow that f is uniformly homotopic to $p_3 \circ \lambda_3 \circ \lambda_2 \circ \lambda_1$. Since $\lambda_3 \circ \lambda_2 \circ \lambda_1$ is a continuous homomorphism of the real line into the multiplicative group of complex numbers of absolute value one, there will have to be a complex number ω such that $\lambda_3(\lambda_2(\lambda_1(t))) = e^{i\omega t}$. Thus we will know that $f(t)$ is uniformly homotopic to $p_3(e^{i\omega t})$. Since $p_3(e^{i\omega t})$ is periodic, this will prove our theorem.

First of all let $x_1, \ldots, x_n$ be the coordinates in the Euclidean space R^n in which X is imbedded. Each Bohr almost periodic real valued function $x_i \circ i \circ f$ has associated with it a Fourier series, $\sum a^i_\alpha e^{i\omega_\alpha t}$. Let M be the smallest subgroup of the real line containing all the ω_α. Consider the Pontrjagin dual M^* of the discrete group M; that is, the collection of all homomorphisms of M into the multiplicative group S^1 of complex numbers of absolute value one. This is a group under pointwise multiplication of functions; one puts the compact open topology on it making it a compact abelian group. For each $t \in R^1$, the map sending $\omega \in M$ into $e^{i\omega t}$ is a homomorphism of M into S^1; thus we get a function from R^1 into M^*. This map is a continuous homomorphism λ_1 of the additive group R^1 onto a dense subgroup of M^*. As is well known, when one makes this standard construction there exists a unique set of continuous functions $\overline{x}_1, \ldots, \overline{x}_n$ of M^* into R^1 such that for each i, $\overline{x}_i(\lambda_1(t)) = x_i(i(f(t)))$.

Since M^* is a compact connected abelian group there exist arbitrarily small closed subgroups A_α such that the quotient groups M^*/A_α are tori.

For each i and any α, let $\overline{\overline{x}}_i(g)$ be $\int_{A_\alpha} \overline{x}_i(g+h)\, d\mu_\alpha(h)$ where μ_α is the normalized Haar measure on A_α. If we take A_α to lie in a sufficiently small neighborhood of the identity in M^* we can make each $\overline{\overline{x}}_i(g)$ approximate as closely as we like to $\overline{x}_i(g)$. By making this approximation sufficiently close we can ensure that the function sending $g \in M^*$ into $(\overline{\overline{x}}_1(g), \ldots, \overline{\overline{x}}_n(g))$ in R^n has an image lying in the open set O which retracts via R onto $i(X)$. This is so because the image of the map sending $g \in M^*$ into $(\overline{x}_i(g), \ldots, \overline{x}_n(g))$ is a compact subset of $i(X)$.

Pick α_0 so that for each $g \in M^*$ not only does $(\overline{\overline{x}}_1(g), \ldots, \overline{\overline{x}}_n(g))$ lie in O but so that the line segment joining it to $(\overline{x}_1(g), \ldots, \overline{x}_n(g))$ lies in O.

Since each function $\overline{\overline{x}}_1(g), \ldots, \overline{\overline{x}}_n(g)$ is constant on cosets $\operatorname{mod} A_{\alpha_0}$, there exist unique continuous functions $y_1, \cdots, y_n$ on M^*/A_{α_0} such that if λ_2 is the natural homomorphism of M^* onto M^*/A_{α_0}, then for each i and each $g \in M^*$, $\overline{\overline{x}}_i(g) = y_i(\lambda_2(g))$. Now let $G(g,s)$ for $g \in M^*$ and $s \in [0,1]$ be defined by $G(g,s) = R((1-s)\overline{x}_1(g) + s\overline{\overline{x}}_1(g), \ldots, (1-s)\overline{x}_n(g) + S\overline{\overline{x}}_n(g))$ where R is the retraction of O onto X. Then $G(g,0) = R(\overline{x}_1(g), \ldots, \overline{x}_n(g))$. Let $R(\overline{x}_1(g), \ldots, \overline{x}_n(g))$ be denoted by $p_1(g)$. Then $p_1 \circ \lambda_1$ equals f. In addition $G(g,1) = R(\overline{\overline{x}}_1(g), \ldots, \overline{\overline{x}}_n(g)) = R(y_i(\lambda_2(g)))$. Let the function sending $\overline{g} \in M^*/A_{\alpha_0}$ into $R(y_i(\overline{g}))$ be denoted by p_2, so that $G(g,1) = p_2(\lambda_2(g))$. Thus p_1 is uniformly homotopic to $p_2 \circ \lambda_2$ and f equals $p_1 \circ \lambda_1$. To complete our proof we need only show that there exist a continuous homomorphism λ_3 of M^*/A_{α_0} into S^1 and a continuous function p_3 of S^1 into X such that $p_3 \circ \lambda_3$ is homotopic (and therefore uniformly homotopic) to p_2.

We are assuming that the fundamental group π of the $K(\pi,1)$ space X has as its only commutative subgroups infinite cyclic groups. Therefore, if e is the identity element of M^*/A_{α_0}, the map of $\pi_1(M^*/A_{\alpha_0}, e)$ into $\pi_1(X, p_2(e))$ induced by p_2 can be factored through a homomorphism of $\pi_1(M^*/A_{\alpha_0}, e)$ into Z, the additive group of the integers. We can identify Z with $\pi_1(S^1, 1)$, and since S^1 is a $K(Z,1)$ space we can realize this homomorphism as that which is induced by a suitable continuous map k of $(M^*/A_{\alpha_0}, e)$ into $(S^1, 1)$. However it is known that every map of a compact abelian group into the multiplicative group S^1 is homotopic to a continuous homomorphism, so our homomorphism of $\pi_1(M^*/A_{\alpha_0}, e)$ into $\pi_1(S^1, 1)$ is that induced by some continuous homomorphism λ_3 of M^*/A_{α_0} into S^1. Thus there is a homomorphism ρ of $\pi_1(S^1, 1)$ into $\pi_1(X, p_2(e))$ such that the homomorphism of $\pi_1(M^*/A_{\alpha_0}, e)$ into $\pi_1(X, p_2(e))$ induced by p_2 is the composition of the homomorphism of $\pi_1(M^*/A_{\alpha_0}, e)$ into $\pi_1(S^1, 1)$ induced by λ_3 with ρ. Since X is a $K(\pi,1)$ space, there is a continuous map p_3 of $(S^1, 1)$ into $(X, p_2(e))$ that induces the homomorphism ρ. Then $p_3 \circ \lambda_3$ induces the same homomorphism of fundamental groups as p_2; since X is a $K(\pi, 1)$ space these maps are homotopic. This concludes the proof of our theorem.

We can get the conclusion of our theorem for certain differentiable manifolds with boundary. For any such manifold M form its double; that is, let $M_1 = \{(p,1)\}$ $(p \in M)$ and $M_2 = \{(p,2)\}$ $(p \in M)$ and form the differentiable manifold N from M_1 and M_2 by identifying $(p,1)$ with $(p,2)$ whenever p belongs to the boundary of M. Let d be the diffeomorphism of N onto itself sending $(p,1)$ to $(p,2)$. We can put a complete metric on N and we will get one that is invariant under d by averaging such a metric with that induced by d.

It is easy to see that if N is a $K(\pi,1)$ space where π is a group whose only abelian subgroups are infinitely cyclic, then the conclusion of our theorem holds for M. In fact let p the projection of N onto M_1 sending (p,i) to $(p,1)$ for $i = 1,2$. If f is a Bohr almost periodic function of the real line into M_1, then there is a uniform homotopy H of f considered as a map of the real line into N to a periodic map of the real line into N. Then since p is uniformly continuous $p \circ H$ gives a uniform homotopy of f in M_1 to a periodic map of the real line into M_1. Since M_1 is homeomorphic to M, we are done. This is the method indicated in [1] for getting the result of our theorem for regions in the plane gotten by punching two or more holes in the interior of a closed disc. (In case there is just one hole the result follows by another argument.)

Let us now suppose that X is a compact Riemannian manifold of everywhere negative sectional curvature. As mentioned in the introduction, X must then be a $K(\pi,1)$ space of the type we are considering.

Theorem 2. *Any Bohr almost periodic geodesic in X must be periodic.*

Proof. Let $f(t)$ be a Bohr almost periodic geodesic; then $f(t)$ is uniformly homotopic to some periodic function $g(t)$. However it is known that any closed curve on X is either homotopic to a constant map or freely homotopic to a unique closed geodesic. Thus there exists a uniformly continuous function F from $R^1 \times [0,1]$ to X such that $F(t,0) = f(t)$ and $F(t,1)$ parametrizes a closed geodesic at a constant rate of speed or else is constant. We can lift F to a map $\widetilde{F}$ of $R^1 \times [0,1]$ into the universal covering space $\widetilde{X}$ of X. The Riemannian metric on X can be lifted uniquely to a Riemannian metric on $\widetilde{X}$. The functions $\tilde{F}(t,0)$ and $\widetilde{F}(t,1)$ will necessarily describe geodesics. (This is so because $\widetilde{F}(t,1)$ cannot be a constant map. In fact, as we will show below, there is a fixed constant which is an upper bound for the distance in $\widetilde{X}$ between $\widetilde{F}(t,0)$ and $\widetilde{F}(t,1)$. Thus if $\widetilde{F}(t,1)$ were constant, $\widetilde{F}(t,0)$ would be a geodesic that stayed in a bounded subset of $\widetilde{X}$. However no geodesic in the covering space of a compact Riemannian manifold of everywhere negative sectional curvature can behave in this way.)

There exists an $r > 0$ such that for any $x \in X$, the open ball B_x^r of radius r about x has the property that its inverse image in $\widetilde{X}$ fibers trivially over B_x^r with a discrete fiber. We can cover X by a finite number $B_1, \ldots, B_k$ of such open balls of radius r. If n is an integer so large that $\frac{1}{n}$ is less than the Lebesgue number of this covering, then any connected set in X of diameter less than $\frac{1}{n}$ must lie in one of the B_i, so any lifting of this set to $\widetilde{X}$ must lie in one component of the inverse image of this B_i. However each component of the inverse image of B_i in $\widetilde{X}$ gets mapped isometrically onto B_i. Thus any two points in the lifting of our connected set are within a distance $\leq 2r$ in $\widetilde{X}$.

Since F is uniformly continuous we can find an integer m so that if $|s_1 - s_2| \leq \frac{1}{m}$ the distance in X from $F(t, s_1)$ to $F(t, s_2)$ is less than $\frac{1}{n}$ for any t.

Then the distance from $\widetilde{F}(t, 0)$ to $\widetilde{F}(t, 1)$ in $\widetilde{X}$ must be less than or equal to $2mr$. Thus we have two geodesics in the covering space of a compact Riemannian manifold of everywhere negative sectional curvature that are always a bounded distance apart. It is known [4] that this implies that they must be the same geodesic. This concludes the proof of our theorem.

If a Bohr almost periodic map f of the real line into a compact Riemannian manifold of everywhere negative sectional curvature were uniformly homotopic to each of two closed geodesics, these two geodesics would be uniformly homotopic to each other. By the same argument as above, the liftings of these two geodesics to $\widetilde{X}$ would stay a bounded distance apart for all time, so that apart from a possible time lag in their relative parametrizations they would be the same geodesic. $\square$

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Department of Mathematics, University of Rhode Island, Kingston, Rhode Island 02881