# ON A NEW CONDITION FOR STRICTLY POSITIVE DEFINITE FUNCTIONS ON SPHERES 

MICHAEL SCHREINER

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#### Abstract

Recently, Xu and Cheney (1992) have proved that if all the Legendre coefficients of a zonal function defined on a sphere are positive then the function is strictly positive definite. It will be shown in this paper that, even if finitely many of the Legendre coefficients are zero, the strict positive definiteness can be assured. The results are based on approximation properties of singular integrals, and provide also a completely different proof of the results of Xu and Cheney.


## 1. Introduction

Let $S_{m} \subset \mathbb{R}^{m+1}$ be the $m$-dimensional unit sphere for $m \geq 1$. A continuous function $K:[-1,1] \rightarrow \mathbb{R}$ defines for a fixed $\eta \in S_{m}$ a so-called $\eta$-zonal function $\xi \mapsto K(\eta \cdot \xi), \xi \in S_{m}$, on the sphere, where • denotes the usual inner product in $\mathbb{R}^{m+1}$. Thus, the function $K(\eta \cdot)$ depends only on the spherical distance $\arccos (\xi \cdot \eta)$ between $\xi$ and $\eta$. Such a continuous function is called positive definite, if for any choice of pairwise distinct points $\eta_{1}, \ldots, \eta_{N} \in S_{m}$, and any non-zero vector $\left(a_{1}, \ldots, a_{N}\right)^{T} \in \mathbb{R}^{N}$,

$$
\begin{equation*}
\sum_{i=1}^{N} \sum_{j=1}^{N} a_{i} a_{j} K\left(\eta_{i} \cdot \eta_{j}\right) \geq 0 \tag{1.1}
\end{equation*}
$$

Schoenberg [8] has shown that if the function $K$ admits the uniformly convergent series expansion

$$
\begin{equation*}
K(t)=\sum_{n=0}^{\infty} k_{n} P_{n}(t), t \in[-1,1] \tag{1.2}
\end{equation*}
$$

in terms of certain Legendre (or Gegenbauer or ultraspherical) polynomials, a sufficient condition for (1.1) is that $k_{n} \geq 0, n=0,1, \ldots$.

However, when dealing with problems of interpolation, a stronger condition on the kernel $K$ is useful: $K$ is called strictly positive definite, if the quadratic form (1.1) is strictly positive for any set $\left\{\eta_{1}, \ldots, \eta_{N}\right\} \subset S_{m}$ of pairwise distinct points and any choice of a non-zero vector $\left(a_{1}, \ldots, a_{n}\right)^{T} \in \mathbb{R}^{N}$. A sufficient condition for strict positive definiteness is, as shown by Xu and Cheney [11], that all the $k_{n}$ in the expansion (1.2) are positive. By a completely different idea we will show in

[^0]this paper that, even if finitely many $k_{n}$ are zero, the kernel $K$ is strictly positive definite.

The value of this result can be explained as follows: assume that a function $F: S_{m} \rightarrow \mathbb{R}$ is known only at finitely many distinct points $\eta_{1}, \ldots, \eta_{N} \in S_{m}$ and one looks for an interpolant of the form

$$
\begin{equation*}
S(\xi)=\sum_{i=1}^{N} a_{i} K\left(\eta_{i} \cdot \xi\right) \tag{1.3}
\end{equation*}
$$

satisfying the interpolation conditions $S\left(\eta_{i}\right)=F\left(\eta_{i}\right), i=1, \ldots, N$. Then the linear system to be solved is

$$
\left(\begin{array}{ccc}
K\left(\eta_{1} \cdot \eta_{1}\right) & \cdots & K\left(\eta_{N} \cdot \eta_{1}\right) \\
\vdots & \ddots & \vdots \\
K\left(\eta_{1} \cdot \eta_{N}\right) & \cdots & K\left(\eta_{N} \cdot \eta_{N}\right)
\end{array}\right)\left(\begin{array}{c}
a_{1} \\
\vdots \\
a_{N}
\end{array}\right)=\left(\begin{array}{c}
F\left(\eta_{1}\right) \\
\vdots \\
F\left(\eta_{N}\right)
\end{array}\right)
$$

If $K$ is strictly positive definite then the matrix is positive definite, i.e. the interpolation problem is solvable for any choice of pairwise distinct nodal points.

In practice, however, there is often more information on the function $F$ available, e.g. the low order moments of $F$ in terms of its spherical harmonic expansion. In this case, it is desirable to use kernels $K$ in (1.3) which are orthogonal to these low order spherical harmonics and to perform the interpolation only for the difference between the function values $F\left(\eta_{i}\right)$ and the values of the known projection of $F$ to the span of the spherical harmonics under consideration. But orthogonality of the kernel $K$ to spherical harmonics means that the corresponding coefficients $k_{n}$ are zero. Thus, the condition given in [11] cannot be satisfied. In this case our stronger result is necessary. An example for such a situation is the approximation of the gravitational potential of the earth. From physical properties and measurements of satellite orbits the contribution of the lower order spherical harmonics are known with a sufficient accuracy, so that e.g. for space borne gradiometry data this approach is very useful, cf. e.g. [4], [7], [9].

The outline of this paper is organized as follows: after some preliminaries, we develop an easy-to-handle equivalent condition for strict positive definiteness. After that, we construct a special strictly positive definite function with vanishing moments, and use afterwards this kernel, which turns out to be a spherical approximate identity, for the proof of our main result.

## 2. Preliminaries

Asume $m \geq 1$ to be a fixed integer for the further investigations. Let $\cdot$ and $|\cdot|$ denote the usual inner product and the Euclidean norm in $\mathbb{R}^{m+1}$, respectively, and let $S_{m}=\left\{\xi \in \mathbb{R}^{m+1}| | \xi \mid=1\right\}$ be the $m$-dimensional unit sphere in $\mathbb{R}^{m+1}$. We write $d \omega_{m}$ for the induced surface element of $S_{m}$. Then it is well-known that the volume of $S_{m}$ is given by

$$
\omega_{m}=\int_{S_{m}} d \omega_{m}=\frac{2 \pi^{\frac{m+1}{2}}}{\Gamma((m+1) / 2)}
$$

We denote the space of continuous respectively square-integrable functions defined on $S_{m}$ by $\mathcal{C}\left(S_{m}\right)$ or $\mathcal{L}^{2}\left(S_{m}\right)$. The standard $\mathcal{L}^{2}\left(S_{m}\right)$-inner product is written as

$$
(F, G)=\int_{S_{m}} F(\xi) G(\xi) d \omega_{m}(\xi), F, G \in \mathcal{L}^{2}\left(S_{m}\right)
$$

In the following, we repeat some basic facts on spherical harmonics and Legendre polynomials. Details can be found e.g. in [6]. Let $\operatorname{Harm}_{n}$ be the space of all spherical harmonics $Y_{n}: S_{m} \rightarrow \mathbb{R}$ of order $n$, i.e. $Y_{n} \in \operatorname{Harm}_{n}$ if and only if $x \mapsto|x|{ }^{n} Y_{n}(x /|x|)$ is a homogeneous harmonic polynomial of degree $n$. The dimension of Harm ${ }_{n}$ is

$$
N(m, n)=\operatorname{dim} \operatorname{Harm}_{n}=\frac{2 n+m-1}{n}\binom{n+m-2}{n-1}
$$

If $Y_{p} \in \operatorname{Harm}_{p}$ and $Y_{q} \in \operatorname{Harm}_{q}$ with $p \neq q$ then $\left(Y_{p}, Y_{q}\right)=0$, i.e. spherical harmonics of different order are orthogonal. We assume that $\left\{Y_{n, 1}, \ldots, Y_{n, N(m, n)}\right\}$ is an orthonormal basis of $\operatorname{Harm}_{n}$. For $p \in \mathbb{N}_{0}$, we let $\operatorname{Harm}_{0, \ldots, p}=\bigoplus_{n=0}^{p} \operatorname{Harm}_{n}$.

The addition theorem for spherical harmonics reads as follows:

$$
\begin{equation*}
\sum_{j=0}^{N(m, n)} Y_{n, j}(\xi) Y_{n, j}(\eta)=\frac{N(m, n)}{\omega_{m}} P_{n}(\xi \cdot \eta), \xi, \eta \in S_{m} \tag{2.1}
\end{equation*}
$$

where $P_{n}$ are the (generalized) Legendre polynomials of degree $n$. They are defined by the requirements

1. $P_{n}$ is a polynomial of degree $n$,
2. $\int_{-1}^{1} t^{l} P_{n}(t)\left(1-t^{2}\right)^{\frac{m-2}{2}} d t=0, l=0, \ldots, n-1$,
3. $P_{n}(1)=1$.

It follows from (2.1) for fixed $\eta \in S_{m}$ that $P_{n}(\cdot \eta) \in \operatorname{Harm}_{n}$ and that for all $Y_{n} \in \operatorname{Harm}_{n}$

$$
Y_{n}(\xi)=\left(Y_{n}, \frac{N(m, n)}{\omega_{m}} P_{n}(\xi \cdot)\right), \xi \in S_{m}
$$

i.e. $\frac{N(m, n)}{\omega_{m}} P_{n}(\cdot)$ is the reproducing kernel of $\operatorname{Harm}_{n}$, cf. [1]. Applying the CauchySchwarz inequality to (2.1), it can be easily deduced that $\left|P_{n}(t)\right| \leq P_{n}(1)=1$ for all $t \in[-1,1]$. Furthermore, the Fourier series of a square-integrable function $F \in \mathcal{L}^{2}\left(S_{m}\right)$ can be written as

$$
\begin{equation*}
F \sim \sum_{n=0}^{\infty} \frac{N(m, n)}{\omega_{m}} \int_{S_{m}} F(\eta) P_{n}(\cdot \eta) d \omega_{m}(\eta) \tag{2.2}
\end{equation*}
$$

If $G \in \mathcal{L}^{2}[-1,1]$, we obtain for fixed $\eta \in S_{m}$ for the $\eta$-zonal function $\xi \mapsto G(\xi \cdot \eta)$, $\xi \in S_{m}$,

$$
\begin{equation*}
\int_{S_{m}} G(\xi \cdot \eta) Y_{n}(\eta) d \omega_{m}=\omega_{m-1} \int_{-1}^{1} G(t) P_{n}(t)\left(1-t^{2}\right)^{\frac{m-2}{2}} d t Y_{n}(\xi), \xi \in S_{m} \tag{2.3}
\end{equation*}
$$

This formula of Funk and Hecke allows to obtain the Fourier coefficients of an $\eta$ zonal function by a simple one-dimensional integration. Together with the addition theorem we conclude that the Fourier expansion of $G(\cdot \eta)$ is given by

$$
\begin{equation*}
G(\cdot \eta) \sim \sum_{n=0}^{\infty} \frac{N(m, n) \omega_{m-1}}{\omega_{m}} \int_{-1}^{1} G(t) P_{n}(t)\left(1-t^{2}\right)^{\frac{m-2}{2}} d t P_{n}(\cdot \eta) \tag{2.4}
\end{equation*}
$$

Note that the Legendre polynomials are also expressible (for $m \geq 2$ ) by the generating function

$$
\frac{1}{\left(1-2 r t+r^{2}\right)^{\frac{m-1}{2}}}=\sum_{n=0}^{\infty}\binom{n+m-2}{n} P_{n}(t) r^{n},|r|<1, t \in[-1,1]
$$

This shows that the Legendre polynomials are up to a constant the Gegenbauer or ultraspherical polynomials $C_{n}^{\lambda}$ with $\lambda=(m-1) / 2$, cf. e.g. [10].

For $r \in(0,1)$ the uniform convergent series

$$
Q_{r}(t)=\sum_{n=0}^{\infty} \frac{N(m, n)}{\omega_{m}} r^{n} P_{n}(t), t \in[-1,1]
$$

has the explicit representation

$$
Q_{r}(t)=\frac{1-r^{2}}{\left(1-2 r t+r^{2}\right)^{\frac{m+1}{2}}}
$$

which is known as Poisson-kernel. The basic property of $Q_{r}$ for our purposes is described in

Theorem 2.1. Let $F: S_{m} \rightarrow \mathbb{R}$ be continuous. Then

$$
\lim _{\substack{r \rightarrow 1 \\ r<1}} \sup _{\xi \in S_{m}}\left|F(\xi)-\int_{S_{m}} Q_{r}(\xi \cdot \eta) F(\eta) d \omega_{m}(\eta)\right|=0
$$

This theorem shows that $Q_{r}$ is an approximate identity in the space $\mathcal{C}\left(S_{m}\right)$, cf. e.g. [2]. We will need a slight generalization of this result. We define for $p \in \mathbb{N}_{0}$

$$
Q_{r}^{0, \ldots, p \perp}(t)=Q_{r}(t)-\sum_{n=0}^{p} \frac{N(m, n)}{\omega_{m}} r^{n} P_{n}(t)
$$

Then it holds obviously $\left(Q_{r}(\eta \cdot), Y_{n}\right)=0$ for all $Y_{n} \in \operatorname{Harm}_{n}, n=0, \ldots, p$. Furthermore, we easily obtain
Corollary 2.2. Let $F \in \mathcal{C}\left(S_{m}\right)$ satisfy $\left(F, Y_{n}\right)=0$ for all $Y_{n} \in \operatorname{Harm}_{n}, n=$ $0, \ldots, p$. Then

$$
\lim _{\substack{r \rightarrow 1 \\ r<1}} \sup _{\xi \in S_{m}}\left|F(\xi)-\int_{S_{m}} Q_{r}^{0, \ldots, p \perp}(\xi \cdot \eta) F(\eta) d \omega_{m}(\eta)\right|=0
$$

## 3. Strictly positive definite kernels

Let $K:[-1,1] \rightarrow \mathbb{R}$ be continuous and assume that $K$ admits the uniformly convergent series expansion

$$
\begin{equation*}
K(t)=\sum_{n=0}^{\infty} k_{n} P_{n}(t) \tag{3.1}
\end{equation*}
$$

with constants $k_{n} \in \mathbb{R}$. Note that $\left|P_{n}(t)\right| \leq P_{n}(1), t \in[-1,1]$, implies that the series (3.1) is absolute and uniformly convergent, if the series $\sum_{n=0}^{\infty}\left|k_{n}\right|$ is convergent. Schoenberg [8] has shown that such a function $K$ is positive definite if $k_{n} \geq 0$ for all $n \in \mathbb{N}$. The strict positive definiteness can be characterized by
Lemma 3.1. Let $K:[-1,1] \rightarrow \mathbb{R}$ be continuous with uniformly convergent series expansion (3.1) and $k_{n} \geq 0$ for all $n \geq 0$. Then $K$ is strictly positive definite if and only if for all pairwisely distinct $\eta_{1}, \ldots, \eta_{N} \in S_{m}$ the functions $K\left(\eta_{1} \cdot\right), \ldots, K\left(\eta_{N} \cdot\right)$ are linearly independent.

Proof. Since $k_{n} \geq 0$ and $\sum_{n=0}^{\infty} k_{n}<\infty$ it follows from [1] that $\tilde{K}: S_{m} \times S_{m} \rightarrow \mathbb{R}$, defined by $\tilde{K}(\xi, \eta)=K(\xi \cdot \eta), \xi, \eta \in S_{m}$, is the reproducing kernel of a Hilbert space $\left(\mathcal{H},(\cdot, \cdot)_{\mathcal{H}}\right)$ with orthonormal basis $\left\{\left(\omega_{m} k_{n} / N(m, n)\right)^{1 / 2} Y_{n, j} \mid n=0, \ldots, j=\right.$
$\left.1, \ldots, N(m, n), k_{n} \neq 0\right\}$. Thus, $K\left(\eta_{i} \cdot \eta_{j}\right)=\left(K\left(\eta_{i} \cdot\right), K\left(\eta_{j} \cdot\right)\right)_{\mathcal{H}}$, and therefore the matrix

$$
\left(\begin{array}{ccc}
K\left(\eta_{1} \cdot \eta_{1}\right) & \cdots & K\left(\eta_{1} \cdot \eta_{N}\right) \\
\vdots & \ddots & \vdots \\
K\left(\eta_{N} \cdot \eta_{1}\right) & \cdots & K\left(\eta_{N} \cdot \eta_{N}\right)
\end{array}\right)
$$

turns out to be a Gram matrix with respect to $K\left(\eta_{1} \cdot\right), \ldots, K\left(\eta_{N^{*}}\right)$, and is therefore positive definite if and only if the functions $K\left(\eta_{1} \cdot\right), \ldots, K\left(\eta_{N^{\cdot}}\right)$ are linearly independent.

## 4. A special strictly positive definite function

In this section it is proved that the function $Q_{r}^{0, \ldots, p \perp}$ is strictly positive definite for any $p \in \mathbb{N}$. We start with

Lemma 4.1. Let $p \in \mathbb{N}_{0}, \eta \in S_{m}$. Then there exists to any $\varepsilon>0$ a continuous function $H: S_{m} \rightarrow \mathbb{R}$ with the following properties:

1. $H(\eta)=1$,
2. $\operatorname{supp} H \subset\left\{\xi \in S_{m}| | \xi-\eta \mid<\varepsilon\right\}$,
3. $\left(H, Y_{n}\right)=0$ for all $Y_{n} \in \operatorname{Harm}_{n}, n=0, \ldots, p$.

Since the proof of this lemma is rather technical we shift it to the end of this paper.

Fundamental is
Theorem 4.2. Let $r \in(0,1), p \in \mathbb{N}_{0}$. Then the function $Q_{r}^{0, \ldots, p \perp}$ is strictly positive definite.
Proof. Let $\eta_{1}, \ldots, \eta_{N} \in S_{m}$ be pairwise distinct. According to Lemma 3.1 we shall show that the functions $Q_{r}^{0, \ldots, p \perp}\left(\eta_{1} \cdot\right), \ldots, Q_{r}^{0, \ldots, p \perp}\left(\eta_{N^{*}}\right)$ are linearly independent. Assume therefore that for $a_{1}, \ldots, a_{N} \in \mathbb{R}$

$$
\sum_{i=1}^{N} a_{i} Q_{r}^{0, \ldots, p \perp}\left(\eta_{i} \cdot\right)=0
$$

Since the series of $Q_{r}^{0, \ldots, p \perp}$ in terms of the Legendre polynomials is uniformly convergent, it follows for all $\xi \in S_{m}, n \geq p+1$,

$$
\begin{aligned}
0 & =\left(\sum_{i=1}^{N} a_{i} Q_{r}^{0, \ldots, p \perp}\left(\eta_{i} \cdot\right), \frac{N(m, n)}{\omega_{m}} P_{n}(\cdot \xi)\right) \\
& =\sum_{i=1}^{N} a_{i} r^{n} P_{n}\left(\eta_{i} \cdot \xi\right) \\
& =r^{n} \sum_{i=1}^{N} a_{i} P_{n}\left(\eta_{i} \cdot \xi\right) .
\end{aligned}
$$

Thus, $\sum_{i=1}^{N} a_{i} P_{n}\left(\eta_{i} \cdot\right)=0$ for all $n \geq p+1$. The same calculation performed in the backward direction implies then

$$
\begin{equation*}
\sum_{i=1}^{N} a_{i} Q_{r}^{0, \ldots, p \perp}\left(\eta_{i} \cdot\right)=0 \tag{4.1}
\end{equation*}
$$

for all $r \in(0,1)$.

Now, let $i^{*} \in\{1, \ldots, N\}$ be fixed, and choose $\varepsilon<\Theta_{N}$, where the nodal width $\Theta_{N}$ is given by

$$
\Theta_{N}=\max _{j=1, \ldots, N} \sup _{j \neq i}\left|\eta_{i}-\eta_{j}\right|
$$

For this $\varepsilon$ and $\eta=\eta_{i^{*}}$ let $H \in \mathcal{C}\left(S_{m}\right)$ have the properties of Lemma 4.1. Then we conclude from Corollary 2.2 and the fact that $H\left(\eta_{i}\right)=0$ if $i \neq i^{*}$,

$$
a_{i^{*}}=\sum_{i=1}^{N} a_{i} H\left(\eta_{i}\right)=\lim _{\substack{r \rightarrow 1 \\ r<1}}\left(\sum_{i=1}^{N} a_{i} Q_{r}^{0, \ldots, p \perp}\left(\eta_{i} \cdot\right), H\right) .
$$

On the other hand side it follows from (4.1) that for all $r \in(0,1)$

$$
\left(\sum_{i=1}^{N} a_{i} Q_{r}^{0, \ldots, p \perp}\left(\eta_{i} \cdot\right), H\right)=0
$$

Thus, $a_{i^{*}}=0$. Since $i^{*} \in\{1, \ldots, N\}$ was chosen arbitrarily, it follows $a_{1}=\ldots=$ $a_{N}=0$, and hence the functions $Q_{r}^{0, \ldots, p \perp}\left(\eta_{1} \cdot\right), \ldots, Q_{r}^{0, \ldots, p \perp}\left(\eta_{N} \cdot\right)$ are linearly independent, as required.

Remark 4.3. It follows by similar (and even easier) arguments that $Q_{r}$ is strictly positive definite. We omit the details.

## 5. The main result

The considerations of the last chapters allow us to prove the main result:
Theorem 5.1. Let $K:[-1,1] \rightarrow \mathbb{R}$ be a continuous kernel with uniformly convergent series expansion

$$
\begin{equation*}
K(t)=\sum_{n=0}^{\infty} k_{n} P_{n}(t) \tag{5.1}
\end{equation*}
$$

Assume that $k_{n} \geq 0, n \in \mathbb{N}$, and only finitely many $k_{n}$ are zero. Then $K$ is strictly positive definite.
Proof. Let $p$ denote the index of the largest integer $n$ for that $k_{n}=0$. Then $k_{n}>0$ for all $n>p$. Assume that $\eta_{1}, \ldots, \eta_{N} \in S_{m}$ are pairwise distinct. We shall show that $K\left(\eta_{1} \cdot\right), \ldots, K\left(\eta_{N^{*}}\right)$ are linearly independent.

If $\sum_{i=1}^{N} a_{i} K\left(\eta_{i^{*}}\right)=0$, similar arguments as in the proof of Theorem 4.2 provide that

$$
\begin{equation*}
\sum_{i=1}^{N} a_{i} P_{n}\left(\eta_{i} \cdot\right)=0 \tag{5.2}
\end{equation*}
$$

for all $n$ for which $k_{n}>0$. In particular, (5.2) is true for all $n>p$. But then it follows that for every $r \in(0,1)$

$$
\sum_{i=1}^{N} a_{i} Q_{r}^{0, \ldots, p \perp}\left(\eta_{i} \cdot\right)=0
$$

We know from Theorem 4.2 that $Q_{r}^{0, \ldots, p \perp}$ is strictly positive definite, therefore $a_{1}=\ldots=a_{N}=0$. Hence, $K$ is strictly positive definite.

Since all the Legendre coefficients of $Q_{r}$ are positive, similar arguments as above together with Remark 4.3 prove the result of Xu and Cheney [11]:

Corollary 5.2. If all the $k_{n}$ in (5.1) are positive then $K$ is strictly positive definite.
The question whether a weaker condition than the one of Theorem 4.2 is sufficient for strict positive definiteness is still open. We show by an easy example that the condition that finitely many $k_{n}$ are greater than zero is not sufficient for strict positive definiteness: let $m=2$ and assume that the $k_{n}$ of the uniformly convergent series expansion of a kernel $K$ are zero for odd $n$ and greater than zero for even $n$. The kernel $K$ is then an even function. If we choose $\eta_{1}$ to be the North Pole and $\eta_{2}$ to be the South Pole, $K$ satisfies $K\left(\eta_{1} \cdot \xi\right)=K\left(\eta_{2} \cdot \xi\right), \xi \in S_{m}$. Thus, $K\left(\eta_{1} \cdot\right)$ and $K\left(\eta_{2} \cdot\right)$ are linearly dependent, and so $K$ is not strictly positive definite.

## Appendix: Proof of Lemma 4.1

Lemma 4.1 follows immediately, if we can construct for any $\beta>0$ and all $p \in \mathbb{N}_{0}$ a continuous function $L:[-1,1] \rightarrow \mathbb{R}$ with the properties

1. $L(1)=1$,
2. $\operatorname{supp} L \subset[1-\beta, 1]$,
3. $\int_{-1}^{1} L(t) P_{n}(t)\left(1-t^{2}\right)^{\frac{m-2}{2}} d t=0, n=0, \ldots, p$,
since then the function $H \in \mathcal{C}\left(S_{m}\right)$ defined by $H(\xi)=L(\eta \cdot \xi), \xi \in S_{m}$, satisfies all requirements of Lemma 4.1 with $\varepsilon=(2 \beta)^{1 / 2}$.

We will show now how a function $L$ satisfying 1 . -3 . can be constructed for a given $\beta>0$ and $p \in \mathbb{N}_{0}$. Assume first that $m$ is even. Choose real numbers $h_{i}$ with $1-\beta \leq h_{0}<h_{1}<\ldots h_{m+p+3}<1$. We define for $t \in[-1,1]$

$$
A_{0}(t)=\frac{\left(t-h_{0}\right)_{+}}{1-h_{0}}
$$

and for any given $h \in(-1,1)$

$$
B_{h}(t)=\left(1-t^{2}\right)(t-h)_{+},
$$

where $t_{+}$means, as usual,

$$
t_{+}=\left\{\begin{array}{lll}
t & \text { for } & t \geq 0 \\
0 & \text { for } & t<0
\end{array}\right.
$$

Setting especially $h=h_{i}, i=1, \ldots, m+p+3$, it follows easily that the functions $A_{0}, B_{h_{i}}, i=1, \ldots, m+p+3$, are linearly independent. Furthermore, they are Lipschitz-continuous functions with $\operatorname{supp} A_{0}=\left[h_{0}, 1\right]$ and $\operatorname{supp} B_{h_{i}}=\left[h_{i}, 1\right], i=$ $1, \ldots, m+p+3$. It can be deduced from the Lipschitz continuity (cf. [5]) that the Fourier series (2.4) of $A_{0}$ and $B_{h_{i}}$ are uniformly convergent, i.e.

$$
B_{h_{i}}(t)=\sum_{n=0}^{\infty} \frac{N(m, n)}{\omega_{m}} B_{h_{i}}^{\wedge}(n) P_{n}(t),
$$

with

$$
B_{h_{i}}^{\wedge}(n)=\omega_{m-1} \int_{-1}^{1} B_{h_{i}}(t) P_{n}(t)\left(1-t^{2}\right)^{\frac{m-2}{2}} d t .
$$

(A similar result holds for $A_{0}$.)
We are looking for a function $L$ of the form

$$
L(t)=A_{0}(t)-\sum_{i=1}^{m+p+3} b_{i} B_{h_{i}}(t), t \in[-1,1]
$$

with parameters $b_{i}$ to be determined. It can be easily deduced that the conditions 1. and 2. are fulfilled for any choice of $b_{1}, \ldots, b_{m+p+3}$. Condition 3. is equivalent to the equations

$$
\begin{equation*}
A_{0}^{\wedge}(n)-\sum_{i=1}^{m+p+3} b_{i} B_{h_{i}}^{\wedge}(n)=0, n=0, \ldots, p \tag{5.3}
\end{equation*}
$$

In order to study these equations, we see first that since $A_{0} \geq 0$ and $A_{0}(1)=1$ it follows that $A_{0}^{\wedge}(0)>0$. Thus, (5.3) is the linear system of equations

$$
\left(\begin{array}{ccc}
B_{h_{1}}^{\wedge}(0) & \cdots & B_{h_{m+p+3}}^{\wedge}(0) \\
\vdots & \ddots & \vdots \\
B_{h_{1}}^{\wedge}(p) & \cdots & B_{h_{m+p+3}}^{\wedge}(p)
\end{array}\right)\left(\begin{array}{c}
b_{1} \\
\vdots \\
b_{m+p+3}
\end{array}\right)=\left(\begin{array}{c}
A_{0}^{\wedge}(0) \\
\vdots \\
A_{0}^{\wedge}(p)
\end{array}\right)
$$

with non-vanishing right hand side. To get more information on the matrix entries, we see that

$$
\begin{aligned}
B_{h}^{\wedge}(n) & =\int_{-1}^{1}(t-h)_{+}\left(1-t^{2}\right) P_{n}(t)\left(1-t^{2}\right)^{\frac{m-2}{2}} d t \\
& =\int_{h}^{1}(t-h) P_{n}(t)\left(1-t^{2}\right)^{\frac{m}{2}} d t
\end{aligned}
$$

is a polynomial in the variable $h$ of degree $m+n+2$. In order to ensure the solvability of the above system, we enlarge it to

$$
\left(\begin{array}{ccc}
1 & \cdots & 1  \tag{5.4}\\
h_{1} & \cdots & h_{m+p+3} \\
\vdots & \ddots & \vdots \\
h_{1}^{m+1} & \cdots & h_{m+p+3}^{m+1} \\
B_{h_{1}}^{\wedge}(0) & \cdots & B_{h_{m+p+3}}^{\wedge}(0) \\
\vdots & \ddots & \vdots \\
B_{h_{1}}^{\wedge}(p) & \cdots & B_{h_{m+p+3}}^{\wedge}(p)
\end{array}\right)\left(\begin{array}{c}
b_{1} \\
\vdots \\
b_{m+p+3}
\end{array}\right)=\left(\begin{array}{c}
0 \\
0 \\
\vdots \\
0 \\
A_{0}^{\wedge}(0) \\
\vdots \\
A_{0}^{\wedge}(p)
\end{array}\right)
$$

This linear system, now, can be seen to come from the one-dimensional polynomial interpolation problem with nodal points $h_{1}, \ldots, h_{m+p+3}$, data values $0, \ldots, 0$, $A_{0}^{\wedge}(0), \ldots, A_{0}^{\wedge}(p)$, and trial functions $\left\{1, h, \ldots, h^{m+1}, B_{h}^{\wedge}(0), \ldots, B_{h}^{\wedge}(p)\right\}$ being polynomials of degrees $0, \ldots, m+p+2$. Since we know from e.g. Davis [3] that this system is unisolvent, there exists a unique solution of (5.4) which also solves then (5.3), and hence fulfills the third requirement.

If $m$ is odd, we can apply the same arguments, but to achieve that the $B_{h}^{\wedge}(n)$ are polynomials in $h$, one has to modify the defining equation of the function $B_{h}$. For odd $m$ the definition

$$
B_{h}(t)=\left(1-t^{2}\right)^{\frac{3}{2}}(t-h)_{+}
$$

will ensure that the above arguments can be applied.

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University of Kaiserslautern, Laboratory of Technomathematics, Geomathematics Group, P.O. Box 30 49, 67653 Kaiserslautern, Germany

E-mail address: schreiner@mathematik.uni-kl.de


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