PROCEEDINGS OF THE
AMERICAN MATHEMATICAL SOCIETY
Volume 125, Number 2, February 1997, Pages 471–475
S 0002-9939(97)03684-8

THE GENERALIZED GOURSAT-DARBOUX PROBLEM FOR A THIRD ORDER OPERATOR

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(Communicated by Jeffrey B. Rauch)

ABSTRACT. It is proved that if a generalized Goursat-Darboux problem is C^∞-well posed then the operator cannot contain derivatives with respect to one of the variables.

1. Introduction

In this paper we study the generalized Goursat-Darboux problem

$$\text{(P)} \qquad \begin{cases} \partial_t\partial_x\partial_y u(t,x,y,z) = (A_1\partial_t^2\partial_z + B_1\partial_x^2\partial_z + C_1\partial_y^2\partial_z \\ \qquad\qquad\qquad + A_2\partial_t^2 + B_2\partial_x^2 + C_2\partial^2 y)u(t,x,y,z), \\ u(0,x,y,z) = g_1(x,y,z), \\ u(t,0,y,z) = g_2(t,y,z), \\ u(t,x,0,z) = g_3(t,x,z). \end{cases}$$

The A_i, B_i, C_i are real numbers and the $g_i, i = 1,2,3$, are given C^∞ functions satisfying the obvious compatibility conditions. We are looking for C^∞ solutions in a neighborhood of the origin of $\mathbf{R}^4$.

The study of this problem faces the usual difficulties of a Goursat problem, but the way the initial conditions are given (with tight compatibility conditions) introduces supplementary difficulties. The Goursat-Darboux problems show up when trying to solve systems of differential equations by the methods of [7] and [8].

Simpler cases of this Goursat-Darboux problem have been studied in [1].

2. Using the Lax-Mizohata theorem

A Lax-Mizohata type theorem is true for the generalized Goursat-Darboux problem [2], and so we can conclude that if the problem (P) is locally C^∞-well posed, the polynomial

$$TXY - A_1T^2Z - B_1X^2Z - C_1Y^2Z$$

can have only real roots in T, X and Y for any values of the other variables.

Received by the editors August 11, 1995.
1991 *Mathematics Subject Classification.* Primary 35L30, 35E15, 35G10.
Key words and phrases. Goursat problem, Darboux problem, hypergeometric functions of several variables, third order operators.
The first author was partially supported by CMUC-Coimbra and JNICT Project no.PMCT/C/CEN/18/90.
The second author was partially supported by CMUC-Coimbra.

It is easy to see that this implies that we must have $A_1C_1 \leq 0$, $A_1B_1 \leq 0$ and $B_1C_1 \leq 0$. This implies that one of A_1, B_1, C_1 must be zero. However, for the moment, we cannot say which one.

3. Using the continuity

If the problem is C^∞-well posed, then it is well known [6] that the map

$$g_1, g_2, g_3 \mapsto u$$

is continuous for the usual topology of C^∞. That implies the inequality

$$|u(t,x,y,z)| \leq c\left[\sup_{\substack{K\\|\alpha|\leq N}} |\partial^\alpha g_1| + \sup_{\substack{K'\\|\alpha|\leq N'}} |\partial^\alpha g_2| + \sup_{\substack{K''\\|\alpha|\leq N''}} |\partial^\alpha g_3|\right] \tag{1}$$

for (t,x,y,z) in a neighborhood of the origin, where K, K' and K'' are compact sets and N, N' and N'' are nonnegative integers. Let's make the separation of variables

$$u(t,x,y,z) = v(t,x,y)e^{i\eta z},$$

where η is a positive real number to be chosen later. Let's choose the initial conditions

$$g_1(x,y,z) = e^{i\eta z}\left(\frac{1}{4}x^2y^2 + \frac{1}{4!}y^4 + \frac{1}{4!}x^4\right),$$
$$g_2(t,y,z) = e^{i\eta z}\left(\frac{1}{4}t^2y^2 + \frac{1}{4!}t^4 + \frac{1}{4!}y^4\right),$$
$$g_3(t,x,z) = e^{i\eta z}\left(\frac{1}{4}t^2x^2 + \frac{1}{4!}t^4 + \frac{1}{4!}x^4\right).$$

Let's look for a solution u of (P) such that v has the form

$$v(t,x,y) = \sum_{p,q,r\geq 0} v_{pqr}\frac{t^p x^q y^r}{p!q!r!}.$$

The initial conditions imply that

$$v_{oqr}, \quad v_{por}, \quad v_{pqo} \tag{2}$$

are all zero, except

$$v_{220}, v_{202}, v_{022}, v_{400}, v_{040}, v_{004}. \tag{3}$$

Let's write $A = A_1(i\eta) + A_2$, $B = B_1(i\eta) + B_2$, $C = C_1(i\eta) + C_2$. Then v must satisfy the differential equation

$$\partial_t\partial_x\partial_y v = (A\partial_t^2 + B\partial_x^2 + C\partial_y^2)v.$$

This implies that the coefficients v_{pqr} satisfy the recurrence relation

$$v_{pqr} = Av_{p+1,q-1,r-1} + Bv_{p-1,q+1,r-1} + Cv_{p-1,q-1,r+1}. \tag{4}$$

To see that this recurrence relation and the conditions (2)–(3) define completely all the v_{pqr}, we define index $(v_{pqr}) = p+q+r$. We see that on the left of (4) the index is one unit more than for each term on the right side. Thus, we can conclude that all coefficients with index 0, 1, 2, 3 and 5 are equal to zero. Those with index 4 that are nonzero are the ones in (3). With index 6 only v_{222} is nonzero. And although some terms with index 7 and 8 are nonzero, only the coefficient v_{333} is nonzero for all v_{pqr} with index 9. So we can write all the v_{pqr} with index greater than 9 as a function of v_{333}.

The following two lemmas are easily proved and tell which of these coefficients are different from zero.

Lemma 1. *Let N be the index of v_{pqr} and $N \geq 9$.*

a) *If N is even, only the v_{pqr} with*

$$p+q+r=N, \qquad p,q,r \leq N-6, \qquad p,q,r \text{ all even},$$

can be different from zero.

b) *If N is odd, only the v_{ppr} with*

$$p+q+r=N, \quad p,q,r \text{ all odd}, \quad p,q,r \leq N-6,$$

can be different from zero.

Lemma 2. *Let N be the index of v_{pqr}, $N \geq 9$, and $p,q,r \leq N-6$. The coefficients v_{pqr} that are different from zero have the form*

$$v_{pqr} = \frac{(p+q+r-9)!}{\alpha!\beta!\gamma!} A^{\alpha} B^{\beta} C^{\gamma} v_{333}$$

with

$$\alpha = \frac{q+r-6}{2}, \quad \beta = \frac{p+r-6}{2}, \quad \gamma = \frac{p+q-6}{2}.$$

Now we can write an explicit formula for $v(t,x,y)$. Modulo polynomials of degree not greater than eight,

$$\begin{aligned} v(t,x,y) = &\sum_{a,b,c\geq 1} v_{2a,2b,2c} \frac{t^{2a}x^{2b}y^{2c}}{(2a)!(2b)!(2c)!} \\ &+ \sum_{a,b,c\geq 0} v_{2a+1,2b+1,2c+1} \frac{t^{2a+1}x^{2b+1}y^{2c+1}}{(2a+1)!(2b+1)!(2c+1)!} \end{aligned}$$

But in this formula there are still coefficients that are zero (and to which we cannot apply the formula of lemma 2). So we must exclude more coefficients and in the end we can write $v(t,x,y)$ as a linear combination of several hypergeometric functions of two or three of the following variables: ACx^2, CAx^2, ABy^2, BAy^2, BCt^2, CBt^2. The important fact is that all the coefficients of the linear combination are different, and so we can control the growth of $v(t,x,y)$ because we can choose t, x or y small enough in a neighborhood of the origin.

To the hypergeometric functions obtained we can apply a slight modification of theorem 2 of [3], to conclude that if all but one variable go to zero when η grows, and the other goes to infinity in absolute value (but is not a negative real number nor is close to negative real number), then each function is asymptotically equivalent to a hypergeometric function of one variable of the type ${}_pF_{p+1}$. In the cases where this function has exponential growth, $v(t,x,y)$ will have exponential growth also, because all the coefficients of the linear combination are different.

Let's do the calculations for one of the hypergeometric functions; for the others the calculations are identical. If $|ACx^2| \underset{\eta\to+\infty}{\longrightarrow} 0$ and $|ABy^2| \underset{\eta\to+\infty}{\longrightarrow} +\infty$ then the hypergeometric function

$$\sum_{b,c\geq 0} \frac{(2b+2c+1)!}{(b+c+1)!(2b+4)!(2c+4)!} \frac{(ACx^2)^b(ABy^2)^c}{b!c!}$$

is asymptotically equivalent to

$$\frac{1}{2\cdot 4!}At^2x^4y^4\sum_{c\geq 0}\frac{(2c+1)!}{(c+1)!(2c+4)!}\frac{(ABy^2)^c}{c!}$$
$$=\frac{1}{2\cdot 4!}At^2x^4y^4\sum_{c\geq 0}\frac{1}{(c+1)!(2c+4)(2c+3)(2c+2)}\frac{(ABy^2)^c}{c!}$$
$$=\frac{1}{16\cdot 4!}At^2x^4y^4\sum_{c\geq 0}\frac{1}{(c+1)!(c+2)(c+\frac{3}{2})(c+1)}\frac{(ABy^2)^c}{c!}.$$

But

$$c+\frac{3}{2}=\frac{\Gamma(c+\frac{5}{2})}{\Gamma(c+\frac{3}{2})}=\frac{(\frac{5}{2})_c\Gamma(\frac{5}{2})}{(\frac{3}{2})_c\Gamma(\frac{3}{2})}$$

where $(a)_c = a(a+1)(a+2)\cdots(a+c-1)$ is the Pochhammer notation. So the function above is

$$\frac{\Gamma(\frac{3}{2})}{\Gamma(\frac{5}{2})}\frac{1}{32\cdot 4!}At^2x^4y^4\sum_{c\geq 0}\frac{(\frac{3}{2})_c(1)_c}{(3)_c(\frac{5}{2})_c(2)_c}\frac{(ABy^2)^c}{c!}$$
$$=\frac{\Gamma(\frac{3}{2})}{\Gamma(\frac{5}{2})}\frac{1}{32\cdot 4!}At^2x^4y^4{}_2F_3\left(\begin{matrix}\frac{3}{2} & & 1 & \\ 3 & \frac{5}{2} & & 2\end{matrix}\middle| ABy^2\right).$$

By Meijer's theorem [5] this hypergeometric function is asymptotically equivalent to

$$\frac{1}{32\cdot 4!\sqrt{\pi}}At^2x^4y^2(ABy^2)^{-9/4}e^{2\sqrt{ABy^2}}.$$

But

$$AB = -A_1B_1\eta^2 + i(A_1B_2 + B_1A_2)\eta + A_2B_2. \tag{5}$$

So, if $-A_1B_1$ is positive, this function will have an exponential growth, and this contradicts the inequality (1). Exchanging the roles of t, x and y we would obtain similarly

$$A_1B_1 \geq 0,$$
$$A_1C_1 \geq 0,$$
$$B_1C_1 \geq 0.$$

4. Conclusion

The conclusions of sections 2 and 3 imply that $A_1B_1 = B_1C_1 = A_1C_1 = 0$, and so at least two of A_1, B_1, C_1 must be zero. Let's assume $B_1 = C_1 = 0$. Then (5) becomes $AB = iA_1B_2\eta + A_2B_2$

But in this case

$$e^{2\sqrt{ABy^2}}$$

will also have an exponential growth, and in the same way as before we can conclude that $A_1B_2 = 0$. Similarly, we would conclude $A_1C_2 = 0$. So, if $A_1 \neq 0$, then $B_2 = C_2 = 0$.

Let's now study the case when $B = C = 0$ and $A \neq 0$. The equation is reduced to

$$\partial_t \partial_x \partial_y u(t,x,y,z) = (A_1 \partial_z + A_2)\partial_t^2 u(t,x,y,z).$$

Let's separate variables again:

$$u(t,x,y,z) = w(x,y,z)(1-e^t),$$

and choose

$$\begin{aligned} &S_1(x,y,z) = 0,\\ &S_2(t,y,z) = (1-e^t)G_2(y,z),\\ &S_3(t,x,z) = (1-e^t)G_3(x,z). \end{aligned}$$

Then we obtain

$$\begin{aligned} &\partial_x \partial_y w(x,y,z) = (A_1 \partial_z + A_2) w(x,y,z),\\ &w(0,y,z) = G_2(y,z),\\ &w(x,0,z) = G_3(x,z). \end{aligned}$$

and we are in the case of Hasegawa [4] and so can conclude that $A_1 = 0$. We then obtain

Theorem. *If problem* (P) *is C^∞-well posed then $A_1 = B_1 = C_1 = 0$, i.e., none of the derivatives with respect to z can appear in the equation.*

This result is in accordance with what was obtained for other cases of the generalized Goursat-Darboux problem ([1] and [4]).

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