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CHERN CHARACTERS ASSOCIATED WITH ALMOST COMMUTING ALGEBRAS

DAOXING XIA

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Abstract. A trace formula related to p-almost commuting subalgebras X and Y is established. By means of this formula, homomorphisms from $K_0(X)$ to $H_\lambda^{\text{odd}}(Y)$ and from $K_1(X)$ to $H_\lambda^{\text{even}}(Y)$ are established. An index map from $K_0(X)\times K_1(Y)$ to $\mathbb{Z}$ is also given.

1. Introduction

Let $\mathcal{A}$ be an algebra over $\mathbb{C}$ with trace ideal J and trace τ. Assume X and Y are two subalgebras of $\mathcal{A}$ satisfying the condition that there is a natural number p such that

$$[x^1,y^1]\cdots[x^p,y^p]\in J$$

for $x^j\in X$ and $y^j\in Y$, where $[x,y]$ is the commutator $xy-yx$. Then we say that X and Y are *p-almost commuting*. The present note is a continuation of [6], [7], [8], [9] to study the Chern characters associated with this pair of X and Y in the context of A. Connes' theory of noncommutative geometry. First in §2, we establish a trace formula (8) of

$$\xi_n(x^0,\dots,x^n;y^0,\dots,y^{n-1})\overset{\text{def}}{=}\tau x^0[x^1,y^0]\cdots[x^n,y^n],\qquad n\ge p,$$

which gives the relation of the cyclic cochains $A_xA_y\xi_k$ and $A_xA_y\xi_{k+2m}$ in terms of cyclic cohomology (see Theorem 1). As a corollary, it reduces to the trace formula of

$$\varphi_n(x^0,\dots,x^n;y^0,\dots,y^n)=\tau[x^0,y^0]\cdots[x^n,y^n],\qquad n\ge p-1,$$

in [7] and [8].

Based on Theorem 1 and A. Connes' theory, in Theorem 2 and Theorem 3 we establish homomorphisms from $K_0(X)$ to $H_\lambda^{\text{odd}}(Y)$ and $K_1(X)$ to $H_\lambda^{\text{even}}(Y)$ by means of Chern characters. But for convenience, in the statement of Theorem 3, we switch the roles of X and Y. In Theorem 4, we give the index map from $K_0(X)\times K_1(Y)$ to $\mathbb{Z}$ in the case when $\mathcal{A}$ is an algebra of operators on a separable complex Hilbert space. We also give a simple example to show that the index map is not trivial.

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In §4, we generalize Theorem 1 to the q-deformed (or q-twisted) commutator case (see Theorem 1′). Besides, we give a theorem (Theorem 5) of Chern character of odd dimension which is just a remark of the previous work in [9].

2. Basic formulas

Let X be an algebra over $\mathbb{C}$. Let $C^n = C^n(X)$ be the space of multilinear functions $f_n(x^0, \dots, x^n)$, $x^j \in X$, let $t : C^n \to C^n$ be the operation

$$\big(tf_n\big)(x^0, \dots, x^n) = (-1)^n f_n(x^n, x^0, \dots, x^{n-1}),$$

let $C^n_\lambda = \{f \in C^n : tf = f\}$, and let $Af = (1 + t + \cdots + t^n)f$, $f \in C^n$. Let b be the Hochshild operation $C^n \to C^{n+1}$,

$$\big(bf\big)(x^0, \dots, x^{n+1}) = \big(b'f\big)(x^0, \dots, x^{n+1}) + (-1)^{n+1} f(x^{n+1}x^0, \dots, x^n),$$

where $\big(b'f\big)(x^0, \dots, x^{n+1}) = \sum_{j=0}^n (-1)^j f_n(x^0, \dots, x^j x^{j+1}, \dots, x^n)$. Let p be the operation $pf = \sum_{j=0}^n (n-j)t^j f$, $f \in C^n$, where $t^0 = 1$. Define $S = 2\pi i bpb'$. Then this operator S coincides with A. Connes' operator [1] S at $Z^n_\lambda = \{f \in C^n_\lambda : bf = 0\}$. Let q be the operation [8] $qf = nAf/2 - ptf$, $f \in C^n$. Let r be the operation [8] $rf = r_n(t)f$, $f \in C^n$, for $n > 0$, where $r_n(\cdot)$ is a polynomial of degree $\le n-1$ satisfying $qf = (1-t)rf$, $f \in C^n$. For convenience, define $\hat{S} = bpb'$ (which is the operator S in [8] and [9]).

The following well-known identities will be needed. Notice that $(1-t)b = b'(1-t)$ (cf. [4], [1]) and that $p(1-t)f = (n+1-A)f$, for $f \in C^n$ (cf. [8]). Hence

$$\hat{S}(1-t) = -bAb \tag{1}$$

and

$$(1-t)\hat{S} = -b'bA. \tag{2}$$

If $f \in C^n_\lambda$, then $Sbf \in C^{n+3}_\lambda$ and (cf. [8])

$$\hat{S}bf = \frac{1}{n+1} bA\hat{S}f, \qquad f \in C^n_\lambda. \tag{3}$$

In the present note, the tensor product spaces $C^n(X) \otimes C^n(Y)$ and $C^n_\lambda(X) \otimes C^n_\lambda(Y)$ of functions of several variables $x^i \in X$ and $y^j \in Y$ are considered. The operations t_x, b_x, b'_x, $p_x, \dots$ or t_y, b_y, b'_y, $p_y, \dots$ are the corresponding operations t, b, b', $p, \dots$ with respect to variables x's and y's respectively.

In §2 and §3, assume that X and Y are two p-almost commuting subalgebras of an algebra $\mathcal{A}$ over $\mathbb{C}$ with trace ideal J and trace τ. Define for $n \ge p$

$$\begin{aligned}
\psi_n(x^0, \dots, x^n; y^0, \dots, y^n) &= \tau x^0 y^0 [x^1, y^1] \cdots [x^n, y^n], \\
\phi_{n-1}(x^0, \dots, x^{n-1}; y^0, \dots, y^{n-1}) &= \tau [x^0, y^0] \cdots [x^{n-1}, y^{n-1}], \\
\xi_n(x^0, \dots, x^n; y^0, \dots, y^{n-1}) &= \tau x^0 [x^1, y^0] \cdots [x^n, y^{n-1}], \\
\eta_n(x^0, \dots, x^{n-1}; y^0, \dots, y^n) &= \tau y^0 [x^0, y^1] \cdots [x^{n-1}, y^n].
\end{aligned}$$

The following is a special case of Lemma 1 in [9] (or [7], [8]).

Lemma 1. *For $n \geq p$,*

$$\xi_{n+1} = b_x \psi_n, \qquad \eta_{n+1} = -t_x b_y \psi_n, \tag{4}$$

$$(1 - t_x)\xi_n = b'_x \phi_{n-1}, \qquad (1 - t_y)\eta_n = b'_y \phi_{n-1}, \tag{5}$$

$$(1 - t_y^{-1})\xi_n = b_x \phi_{n-1}, \qquad (1 - t_x^{-1})\eta_n = b_y \phi_{n-1}, \tag{6}$$

$$(1 - t_y)\psi_n = b'_y \xi_n - t_y \phi_n, \qquad (1 - t_x)\psi_n = b'_x \eta_n + \phi_n. \tag{7}$$

Theorem 1. *Let $k \geq p$ be an even number, m be a natural number, and $n = k+2m$. Then there are $\theta_j \in C_\lambda^j(X) \otimes C_\lambda^j(Y)$, $j = n-1, n-2$ (θ_j also depending on k), such that*

$$A_x A_y \xi_n = b_x \theta_{n-1} + A_x S_x b_y \theta_{n-2} + \tilde{\xi}_n, \tag{8}$$

where

$$\tilde{\xi}_n = \frac{1}{(2\pi)^{2m}} \frac{k!(k-1)!}{n!(n-1)!} (A_x S_x)^m (A_y S_y)^m A_x A_y \xi_k. \tag{9}$$

Proof. Based on Lemma 1, in [8] (cf. (20) of [8]) it is proved that

$$A_x A_y \psi_j = (j+1)^2 \psi_j - (j+1) p_y b'_y \xi_j - p_x b'_x A_y \eta_j - (j+1) q_y \phi_j, \tag{10}$$

for $j \geq p$. From (4), (10) and the fact that $b_x \xi_j = 0$, we have

$$\xi_{j+1} = \frac{1}{(j+1)^2}(b_x A_x A_y \psi_j + \hat{S}_x A_y \eta_j) + \frac{1}{j+1} b_x q_y \phi_j, \tag{11}$$

for $j \geq p$. Thus

$$A_x A_y \xi_{j+1} = \frac{j+2}{j+1} b_x A_x A_y \psi_j + \frac{1}{j+1} A_x \hat{S}_x A_y \eta_j, \tag{12}$$

for $j \geq p$, since $A_y q_y = 0$.

Similarly, we have

$$\eta_{j+1} = -\frac{1}{(j+1)^2}(A_x b_y A_y \psi_j + A_x \hat{S}_y \xi_j) - \frac{1}{j+1} t_x b_y q_y \phi_j, \tag{13}$$

for $j \geq p$. It is easy to see that $r_x \phi_j = r_y \phi_j$, since $t_x t_y \phi_j = \phi_j$. Denote $r_x \phi_j$ by $r\phi_j$ (cf. [8]). From (1), we have

$$\begin{aligned} \hat{S}_x t_x q_y \phi_j &= \hat{S}_x t_x (1 - t_y) r_y \phi_j = -\hat{S}_x (1 - t_x) r_y \phi_j \\ &= b_x A_x b_x r \phi_j, \end{aligned} \tag{14}$$

for $j \geq p-1$. From (6), it is easy to see that $t_y^l \xi_j = \xi_j + \sum_{v=1}^{l} t_y^v b_x \phi_{j-1}$. Thus

$$A_y \xi_j = j \xi_j + b_x p_y t_y \phi_{j-1}, \qquad j \geq p. \tag{15}$$

Similar to (14), we have $\hat{S}_y q_y \phi_j = -b_y A_y b_y r \phi_j$. From (7), $A_y \phi_j = b_y A_y \xi_j$. Hence $b_x A_y \phi_j = 0$. Thus

$$b_x \hat{S}_y t_y p_y \phi_j = b_x \hat{S}_y (j A_y \phi_j / 2 - q_y \phi_j) = b_x b_y A_y b_y r \phi_j. \tag{16}$$

From (12), (13), (14), (15) and (16), we have, for $j \geq p$,

$$\begin{aligned} A_x A_y \xi_{j+2} = &\frac{j+3}{j+2} b_x A_x A_y (\psi_{j+1} - \frac{1}{j+1} b_x b_y r \phi_j) \\ &- \frac{1}{(j+1)^2} A_x \hat{S}_x A_x b_y A_y (\psi_j - \frac{1}{j} b_x b_y r \phi_{j-1}) \\ &- \frac{1}{(j+2)(j+1)^2 j} A_x \hat{S}_x A_x A_y \hat{S}_y A_y \xi_j. \end{aligned} \tag{17}$$

As in [8], define $\Theta_{k-1} = \Theta_{k-2} = 0$,

$$\zeta_j = \psi_j - \frac{1}{j} b_x b_y r \phi_{j-1} - \frac{1}{j(j-1)^2} \hat{S}_x \hat{S}_y \Theta_{j-2}$$

and $\Theta_j = \frac{1}{j+1} A_x A_y \zeta_j$ for $j \geq k$ by mathematical induction. Define

$$\hat{\xi}_j = A_x A_y \xi_j - (j+1) b_x \Theta_{j-1} + \frac{1}{j-1} A_x \hat{S}_x b_y \Theta_{j-2}.$$

From (3) and (17), we have

$$\hat{\xi}_{j+2} = -A_x \hat{S}_x A_y \hat{S}_y \hat{\xi}_j / (j+2)(j+1)^2 j, \qquad j \geq k,$$

which proves (8) and (9) where $\theta_{n-1} = (n+1)\Theta_{n-1}$ and $\theta_{n-2} = -\Theta_{n-2}/(n-1)2\pi i$. $\square$

Since $A_x \phi_n = A_y \phi_n$, denote it by $A\phi_n$. Applying b_y to (8), we get the following.

Corollary 1 [8]**.** *Let $k \geq p$ be an even number, m be a natural number and $n = k + 2m$. Then there is $\Theta_{n-1} \in C_\lambda^{n-1}(X) \otimes C_\lambda^{n-1}(Y)$ such that*

$$A\phi_n = b_x b_y \Theta_{n-1} + \tilde{\phi}_n,$$

where $\tilde{\phi}_n = k! S_x^m S_y^m A\phi_n / (2\pi)^{2m} n!$.

3. Chern characters

Let $\ell \geq 1$. For $(x_{ij}) \in M_\ell(X)$, let $[(x_{ij}), y] = ([x_{ij}, y])$, $y \in Y$, where $M_\ell(X)$ is the algebra of $\ell \times \ell$ matrices over X. Define

$$\big(\operatorname{tr} \sharp \xi_n\big)(x^0, \dots, x^n; y^0, \dots, y^{n-1}) = \big(\operatorname{tr} \sharp \tau\big)(x^0 [x^1, y^0] \cdots [x^n, y^{n-1}]), \tag{18}$$

for $x^j \in M_\ell(X)$, $n \geq p$. For odd $n \geq p-1$ and $e \in \operatorname{Proj}(M_\ell(X))$, let

$$\operatorname{ch}_{e,n}(y^0, \dots, y^n) = \frac{n!}{\left(\frac{n+1}{2}\right)!} (-2\pi i)^{\frac{n+1}{2}} A_y \big(\operatorname{tr} \sharp \xi_{n+1}\big)(e, \dots, e; y^0, \dots, y^n).$$

For $f \in Z_\lambda^n$, let $[f] = f + bC_\lambda^{n-1} \in H_\lambda^n = Z_\lambda^n / bC_\lambda^{n-1}$. Let $H_\lambda^{\mathrm{odd}}(Y)$ be the group defined in [1].

Theorem 2. *The mapping $[e] \mapsto [\operatorname{ch}_{e,2m-1}]$ is a homomorphism from $K_0(X)$ to $H_\lambda^{2m-1}(Y)$, $2m \geq p$. It satisfies*

$$[S \operatorname{ch}_{e,2m-1}] = [\operatorname{ch}_{e,2m+1}], \tag{19}$$

and it defines a homomorphism from $K_0(X)$ to $H_\lambda^{\mathrm{odd}}(Y)$.

Proof. We only have to consider the case $e \in \mathrm{Proj}(X)$. It is easy to see (cf. [1]) that for even n,

$$(bf)(e,\dots,e) = f(e,\dots,e) = 0, \qquad f \in C_\lambda^{n-1}, \tag{20}$$

and

$$(A\hat{S}f)(e,\dots,e) = (n+3)(n+2)f(e,\dots,e)/2, \qquad f \in C^n. \tag{21}$$

By (7), $b_yA_y\xi_n(e,\dots,e;y) = A_y\phi_n(e,\dots,e;y) = A_y\tau[e,y^0]\cdots[e,y^n] = 0$ for odd $n \geq p-1$. Thus $\mathrm{ch}_{e,n} \in Z_\lambda^n(Y)$. On the other hand, $A_x\xi_n(e,\dots,e;y) = (n+1)\xi_n(e,\dots,e;y)$ for even n. From (8), (9), (20) and (21), we have

$$\begin{aligned} A_y\xi_n(e,\dots,e;y^0,\dots,y^{n-1}) &= b_y\big(A_x\hat{S}_x\theta_{n-2}\big)(e,\dots,e;y^0,\dots,y^{n-1}) \\ &\quad + (-1)^m\frac{(k-1)!}{(n-1)!2^m}\big[(A_y\hat{S}_y)^mA_y\xi_k\big](e,\dots,e;y^0,\dots,y^{n-1}), \end{aligned}$$

for $n = 2m+k$ and $k \geq p$, which proves (18). The rest of the proof is similar to the proof of Proposition 14 of Chapter II of [1]. $\square$

Remark. By (12) and (21), for odd $n \geq p-1$, we also have

$$\mathrm{ch}_{e,n}(y^0,\dots,y^n) = \frac{n!(n+2)}{\big(\frac{n+1}{2}\big)!2}(-2\pi i)^{\frac{n+1}{2}}\big(A_y\,\mathrm{tr}\,\sharp\eta_n\big)(e,\dots,e;y^0,\dots,y^n).$$

For $(y_{ij}) \in M_\ell(Y)$, let $[x,(y_{ij})] = ([x,y_{ij}])$, $x \in X$. Define $\mathrm{tr}\,\sharp\xi_n$ by (18) for $y^j \in M_\ell(Y)$ and $n \geq p$. For $u \in GL_\ell(Y)$ and even $n \geq p$, define

$$\mathrm{ch}_{u,n}(x^0,\dots,x^n) = k_n\big(A_xA_y\,\mathrm{tr}\,\sharp\xi_n\big)(x^0,\dots,x^n;u^{-1},u,\dots,u^{-1},u), \tag{22}$$

where $k_n = (-2\pi i)^{n/2}n!/(n+1)(\frac{n}{2})!2^{\frac{n}{2}}$.

Theorem 3. *The mapping* $[u] \mapsto [\mathrm{ch}_{u,2m}]$ *is a homomorphism from* $K_1(Y)$ *to* $H_\lambda^{2m}(X)$*, for* $2m \geq p$*. It satisfies*

$$[S\,\mathrm{ch}_{u,2m}] = [\mathrm{ch}_{u,2(m+1)}] \tag{23}$$

and it defines a homomorphism from $K_1(Y)$ *to* $H_\lambda^{\mathrm{even}}(X)$*.*

Proof. We only have to consider the case that $u \in Y$ and $u^{-1} \in Y$. It is easy to see that

$$\big(A\phi_{2m-1}\big)(x^0,\dots,x^{2m-1};u^{-1},u,\dots,u^{-1},u) = 0. \tag{24}$$

For $f \in C^n$, $n = 2m$,

$$bAf = Abf - bf + \sum_{j=1}^{n}(-1)^{j-1}\big(f - t^jf\big)(a^ja^{j+1},\dots,a^{n+1},a^0,\dots). \tag{25}$$

By (4) and (7) we have

$$(1-t_x)\xi_n = (1-t_x)b_x\psi_{n-1} = b_x'(1-t_x)\psi_{n-1} = b_x'\phi_{n-1}, \qquad n \geq p+1.$$

From (24), it follows that $(1-t_x)A_y\xi_{2m}(x^0,\dots,x^n;u^{-1},u,\dots,u^{-1},u) = 0$. By $b_x\xi_n = 0$ and (25), we have

$$\big(b_xA_xA_y\xi_{2m}\big)(x^0,\dots,x^{2m+1};u^{-1},u,\dots,u^{-1},u) = 0.$$

Thus $\mathrm{ch}_{u,2m} \in Z_\lambda^{2m}$. Formula (23) comes from Theorem 1. Let us follow the lines of the proof of Proposition 15 of Chapter II of [1]. It is easy to see that

$$\xi_n(x^0, \dots, x^n; y^0, \dots, y^{n-1}) = 0$$

if one of the y's is 1. Thus in (22), u^{-1} and u may be replaced by $u^{-1}-1$ and $u-1$ respectively.

For $f \in C_\lambda^{2m-1}$, it is easy to calculate that

$$\big(A\hat{S}f\big)(u^{-1}, u, \dots, u^{-1}, u) = 4(m+1)mf(u^{-1}, u, \dots, u^{-1}, u).$$

Similar to the proof of Proposition 15 of Chapter II of [1], we may prove that $\mathrm{ch}_{uv,2m} - \mathrm{ch}_{u,2m} - \mathrm{ch}_{v,2m}$ is the boundary of a cyclic cochain in $C_\lambda^{2m-1}(X)$ since

$$b_y A_y \xi_n = A\phi_n = -b_x A_x \eta_n$$

by (7), which proves the theorem. $\square$

Remark. Similar to (15), by (6), we have $A_x\eta_m = m\eta_m + t_x p_x b_y \phi_{m-1}$. Suppose n is even and $\geq p+1$. By (24), we have

$$\begin{aligned}\big(A_x\eta_{n-1}\big)&(x^0, \dots, x^{n-2}; u^{-1}-1, u-1, \dots, u^{-1}-1, u-1)\\ &= (n-1)\eta_{n-1}(x^0, \dots, x^{n-2}; u^{-1}-1, u-1, \dots, u^{-1}-1, u-1).\end{aligned}$$

By (12), we may prove that

$$\Big(A_xA_y\xi_n - \frac{(n+1)}{n(n-1)}\hat{S}_xA_xA_y\eta_{n-1}\Big)(\cdot\,; u^{-1}-1, \dots, u-1) \in bC_\lambda^{n-1}(X).$$

Thus for even $n \geq p-1$, we may define

$$\begin{aligned}&\mathrm{ch}_{u,n}(x^0, \dots, x^n)\\ &\quad= \hat{k}_n\big(A_xA_y\,\mathrm{tr}\,\sharp\eta_{n+1}\big)(x^0, \dots, x^n; u^{-1}-1, u-1, \dots, u^{-1}-1, u-1),\end{aligned}$$

where $\hat{k}_n = -(-2\pi i)^{n/2} n!/\left(\frac{n+2}{2}\right)!2^{\frac{n+2}{2}}$.

Now let us consider the case that X and Y are p-almost commuting subalgebras of the operator algebra on a complex separable Hilbert space $\mathcal{H}$ and τ is the usual trace. In this case, the elements of $M_\ell(X)$ or $M_\ell(Y)$ may be regarded as operators on $\mathcal{H} \otimes \mathbb{C}^\ell$.

Theorem 4. *The index map*

$$\begin{aligned}\langle [e], [u]\rangle &= \frac{1}{m}(-1)^m\big(A_y\,\mathrm{tr}\,\sharp\xi_{2m}\big)(e, \dots, e; u^{-1}, u, \dots, u^{-1}, u)\\ &= \mathrm{index}\, eu\big|_{\mathrm{range}(e)},\end{aligned}$$

where $e \in \mathrm{Proj}\, M_\ell(X)$ and $u \in GL_\ell(Y)$, is a homomorphism from $K_0(X) \times K_1(Y)$ to $\mathbb{Z}$.

Proof. It is easy to calculate that

$$\langle [e], [u]\rangle = \mathrm{trace}\big((I-ca)^m - (I-ac)^m\big),$$

where $a = eu\big|_{\mathrm{range}(e)}$ and $c = eu^{-1}\big|_{\mathrm{range}(e)}$. The rest of the proof is similar to the corresponding part of Chapter I of [1]. $\square$

Example. Suppose e is the orthogonal projection from $L^2(\mathbb{T})$ to $H^2(\mathbb{T})$, and u is the backward bilateral shift $\big(uf\big)(z) = \overline{z}f(z)$. Then $\langle [e], [u]\rangle = 1$.

4. Deformed commutator case

Let $\mathcal{A}$ be an algebra over $\mathbb{C}$ with trace ideal J and trace τ. Let X and Y be subgroups of $\mathcal{A}$. Assume that there is a function $q : X \times Y \to \mathbb{C}$ satisfying

$$q(x^1x^2; y^1y^2) = \prod_{i,j=1}^{2} q(x^i, y^j) \quad \text{and} \quad q(x,1) = q(1,y) = 1$$

for $x^i, x \in X$ and $y^j, y \in Y$. Let $\{x, y\} = xy - q(x,y)yx$ be the q-deformed commutator. Assume that there is a natural number p such that

$$\{x^1, y^1\} \cdots \{x^p, y^p\} \in J.$$

(We say that X and Y are p-almost q-deformed commuting.) Let

$$M^{m,n} = \bigl\{(x^0, \dots, x^m; y^0, \dots, y^n) \in X^{m+1} \times Y^{n+1} : q(x^0 \cdots x^m, y^0 \cdots y^n) = 1\bigr\}.$$

By means of the operation defined in [9], we may generalize Theorem 1 to the following.

Theorem 1′. *Let k be an even number, $k \geq p$, $m \in \mathbb{N}$, and $n = k+2m$. Then there exists $\theta_j = \theta_j(x^0, \dots, x^j; y^0, \dots, y^j)$ satisfying $\tau_x\theta_j = \tau_y\theta_j$ on $M^{j,j}$ for $j = n-1$ and $n-2$ such that*

$$\alpha_x\alpha_y\xi_n = \delta_x\theta_{n-1} - \alpha_x S_x \delta_y \theta_{n-2} + \tilde{\xi}_n \quad \text{on } M^{n,n-1},$$

where

$$\tilde{\xi}_n = \frac{1}{(2\pi)^{2m}} \frac{k!(k-1)!}{n!(n-1)!} (\alpha_x S_x)^m (\alpha_y S_y)^m \alpha_x \alpha_y \xi_k.$$

Let $Q = \{q(x,y) : x \in X, y \in Y\}$. Let $\mathcal{W} = \{(x,y,c) : x \in X, y \in Y, c \in Q\}$ be the group with multiplication $(x^0, y^0, c^0)(x^1, y^1, c^1) = \bigl(x^0x^1, y^0y^1, c^0c^1q(x^0, y^1)\bigr)$. Let $p : \mathcal{W} \to \mathcal{A}$ be the mapping $p(x,y,c) = cyx$. Then the "curvature" of this mapping is defined as $\omega(w^0, w^1) = p(w^0w^1) - p(w^0)p(w^1)$, $w^0, w^1 \in \mathcal{W}$. For $n \geq p$, define the Chern character of dimension $2n-1$ (see [3] and [5]) as

$$\begin{aligned} &\mathrm{ch}_{2n-1}(w^0, \dots, w^{2n-1}) \\ &\quad = \tau\bigl(\omega(w^0, w^1) \cdots \omega(w^{2n-2}, w^{2n-1}) - \omega(w^{2n-1}, w^0) \cdots \omega(w^{2n-3}, w^{2n-2})\bigr). \end{aligned}$$

Theorem 5. *If $p = 1$ or 2, then there are $(2n-2)$-cyclic cochains f_{2n-2} such that*

$$\mathrm{ch}_{2n-1} = b f_{2n-2} \qquad \text{off } M^{2n-1,2n-1} \tag{26}$$

for $n \geq p$.

Proof. By the proof of the corollary of Theorem 2 in [9], it is easy to see that (26) holds for $n = p$. Then the formula (26) for $n > p$ follows from the fact that there is a constant k_n such that ch_{2n+1} and $k_n S\,\mathrm{ch}_{2n-1}$ are in the same cohomology class by Proposition 15 of Chapter II of [1]. ∎

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Department of Mathematics, Vanderbilt University, Nashville, Tennessee 37240
E-mail address: xiad@ctrvax.vanderbilt.edu