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RADIAL LIMIT OF LACUNARY FOURIER SERIES WITH COEFFICIENTS IN NON-COMMUTATIVE SYMMETRIC SPACES

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ABSTRACT. Let E be a rearrangement invariant space, $\Lambda \subseteq \mathbb{Z}$ an arbitrary set and (M,τ) a von Neumann algebra with a semifinite normal faithful trace. It is proved that the associated symmetric space of measurable operators $E(M,\tau)$ has Λ-RNP if and only if E has Λ-RNP extending in this way some previous results by Q. Xu.

1. INTRODUCTION

The aim of the present note is to solve a problem concerning the Λ-Radon Nikodym property (in short Λ-RNP) in symmetric spaces of measurable operators. For stating our result we shall introduce first the necessary notations and definitions. Let $\Lambda \subseteq \mathbb{Z}$ be an arbitrary set. A Banach space B is said to have Λ-RNP if and only if every B-valued bounded lacunary Fourier series $f(re^{it}) = \sum_{n\in\Lambda} a_n r^{|n|} e^{int}$ on the unit disc D in the complex plane has radial limit at the boundary almost everywhere. As was proved in the paper of Buchvalov and Danilevich [BD] we have $\mathbb{Z}$-RNP=RNP, the usual Radon Nikodym property and also $\mathbb{N}$-RNP=ARNP, the so-called analytic Radon Nikodym property. We consider now (M,τ) a semifinite von Neumann algebra acting on a Hilbert space H, with a normal faithful trace τ. Let $\overline{M}$ be the space of all measurable operators with respect to (M,τ) in the sense of [N] equipped with the measure topology defined there. For $a \in \overline{M}$ and $t > 0$ the t-th singular number of a is defined by (cf. [FK])

$$\mu_t(a) = \inf\{\, \|ae\| \,;\, e \text{ is a projection in } M \,,\ \tau(1-e) \le t \,\}.$$

The function $t \to \mu_t(a)$ will be denoted by $\mu(a)$. For the main properties of this function the reader is referred to [FK]. Let also E be a rearrangement invariant (r.i.) function space on $(0,\infty)$ (cf. [LT]). We define the non-commutative symmetric space associated with (M,τ) and E as follows (cf. [DDP1]):

$$E(M,\tau) = \{\, a \in \overline{M}\,;\, \mu(a) \in E \,\},$$

$$\|a\|_{E(M,\tau)} = \|a\|_E = \|\mu(a)\|_E,\ a \in E(M,\tau).$$

More exactly, we shall consider throughout this paper the following two cases in order to preserve the main situations from the commutative case. If (M,τ) is

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diffuse (cf. [SZ]) then E is taken to be a r.i. space on $[0, \tau(1))$ and when (M, τ) is completely atomic with all the minimal projections of the same trace, E will be a r.i. sequence space (cf. [LT]).

The problem we are mainly concerned here is if Λ-RNP for E is equivalent with Λ-RNP for $E(M, \tau)$. Actually, in this paper we settle this question in the affirmative. In the particular cases $\Lambda = \mathbb{Z}$ and $\Lambda = \mathbb{N}$ this study was done in [X2] and [X1] respectively. In [X1] the author also considered some other cases such as uniform convexity (see also [M]), uniform PL-convexity and uniform H-convexity. Note however that such a result does not hold for AUMD property which does not pass from E to $E(M, \tau)$ even in the particular case of the Schatten class (cf. [HP]). Also there exists E_1 isomorphic with E_2 and such that $E_1(M, \tau)$ is not equal with $E_2(M, \tau)$ (cf. [A] where such an example is given for $M = B(H)$). The arguments from [X1, X2] do not seem to work in this more general case. Our solution is based on some recent results of the author in [M] and also on a compactness theorem in [DDP2].

2. The proof

Let $\mathbf{T}$ be the unit circle equipped with its normalised Haar measure dm. We recall first that a vector measure $F : (T, dm) \to B$ has L^∞-bounded variation (cf. [DU]) if and only if there exists $C > 0$ such that $\|F(A)\|_B \leq C \cdot m(A)$ for every measurable subset A of $\mathbf{T}$. The smallest constant for which the inequality holds is called the L^∞-norm of F. For $\Lambda \subseteq \mathbb{Z}$ we shall denote by $V^\infty_\Lambda(B)$ the space of all vector measures F with L^∞-bounded variation and for which the Fourier coefficients

$$\widehat{F}(n) = \int e^{-int} dm(t) = 0$$

for all $n \in \mathbb{Z} \setminus \Lambda$. F is said to be representable if there exists $f \in L^\infty(B)$ such that $F = f \cdot dm$. Also, we define $H^\infty_\Lambda(B)$ to be the Hardy space of all functions $f : D \to B$ of the form $f(re^{it}) = \sum_{n\in\Lambda} a_n r^{|n|} e^{int}$ endowed with the L^∞-norm. For proving our main result we need the next lemma which can be proved as in [Bl].

Lemma 2.1. *Let $\Lambda \subseteq \mathbb{Z}$ and B be an arbitrary Banach space. Then,*
(a) $H^\infty_\Lambda(B) = V^\infty_\Lambda(B)$ *via the Poisson integral.*
(b) *B has Λ-RNP iff every $F \in V^\infty_\Lambda(B)$ is representable.*

We can present now the main theorem in this paper as follows.

Theorem 2.2. *We consider $\Lambda \subseteq \mathbb{Z}$ as above. A rearrangement invariant space E has Λ-RNP if and only if its non-commutative analogue $E(M, \tau)$ has Λ-RNP.*

Proof. We shall give the proof in the "continuous case" when (M, τ) is diffuse. The "discrete case" can be done in the same way with some minor natural changes. Since E is isometric isomorphic with a closed subspace of $E(M, \tau)$ it is sufficient to prove just "E has Λ-RNP $\Rightarrow E(M, \tau)$ has Λ-RNP". We can assume card $\Lambda = \infty$ (if not, all Banach spaces have Λ-RNP) and remark as in [BD] that E does not contain any copy of c_0. In particular the dual space E^* is equal with the associate space E' (cf. [LT]). We consider first the situation $\tau(1) < \infty$ and we take $F \in V^\infty_\Lambda(E(M, \tau))$. To settle our problem we have to show that F is representable (cf. lemma 2.1) or more particular, ReF is representable ($F = ReF + iImF$). Using a result from [DU] it is also sufficient to prove that for any $A \subseteq T$, $m(A) \neq 0$, there exists $A_0 \subseteq A$, $m(A_0) \neq 0$ such that the set

$$\mathcal{A}(A, A_0) = \{ \frac{ReF(B)}{m(B)} \, ; \, B \subseteq A_0, \; m(B) \neq 0 \} \tag{2.1}$$

is relatively weakly compact or, equivalently, relatively $\sigma(E(M,\tau), E^{'}(M,\tau))$ compact by a result from [DDP2]. Anyway, since (T, dm) is countable generated we deduce (cf. [DU]) that the range of ReF is included into a separable subspace of $E(M,\tau)$. By a density result in [M] this implies that there exists a countable set $\mathcal{C} \subseteq E(M,\tau)$ such that every $T \in \mathcal{C}$ is selfadjoint, has $\mu(T)$ invertible and every element in $\overline{Sp(range(ReF))}$ can be approximated in the norm of $E(M,\tau)$ by a net of measurable operators from $\mathcal{C}$. Let's say $\mathcal{C} = (T_n)_n$. For every $n \in \mathbb{N}$ there exists (cf. [M]) a bounded linear map

$$\Phi_n : E(M,\tau) \to E$$

with $\Phi_n(|T_n|) = \mu(T_n)$ and the norms of Φ_n do not depend on $n \in \mathbb{N}$. Also, we define as in [M]

$$\Psi_n : E \to E(M,\tau),$$

$$\Psi_n(f) = f \circ \mu^{-1}(T_n)(|T_n|)$$

and recall (cf. [M]) that $\mu(\Psi_n(f)) = f^*$, the usual nonincreasing rearrangement of $|f|$ (cf. [LT]). This means in particular that the space $E_n = \Psi_n(E) \subseteq E(M,\tau)$ is in fact a "copy" of E which "stays" on the "direction" of $|T_n|$. After this preparation, we can start the proof of (2.1). So, let's fix $A \subseteq T$, $m(A) \neq 0$ and we want to define A_0. For this, we also consider $n \in \mathbb{N}$ and since E_n has Λ-RNP it follows (cf. lemma 2.1) that the measure $\Psi_n \circ \Phi_n \circ ReF$ is representable. We obtain by a result from [DU] that for every $\epsilon > 0$ and every $C \subseteq T$, $m(C) \neq 0$ there exists $C_0 \subseteq C$, $m(C_0) \neq 0$, $m(C \setminus C_0) < \epsilon$, such that

$$\mathcal{A}^n(C, C_0) = \{ \frac{\Psi_n \circ \Phi_n \circ ReF(D)}{m(D)} \, ; \, D \subseteq C_0, m(D) \neq 0 \} \tag{2.2}$$

is relatively norm compact in E_n. Now, it is obvious to choose a set $A_0 \subseteq A$, $m(A_0) \neq 0$, with the property that for every $n \in \mathbb{N}$ the set $\mathcal{A}^n(A, A_0)$ is relatively norm compact in E_n. We will show that this A_0 is a good set for our problem. To see that $\mathcal{A}(A, A_0)$ is relatively $\sigma(E(M,\tau), E^{'}(M,\tau))$ compact is equivalent (using a result from [DDP2] since $\tau(1) < \infty$) to proving the following assertion:

"For every $x \in E^{'}(M,\tau)$ and $(x_n)_n \subseteq \Omega(x) := \{z \, ; \int_0^t \mu_s(z)ds \leq \int_0^t \mu_s(x)ds\}$ with $x_n \to x$ (μ) we have

$$\sup\{\int_0^\infty \mu_t(x_n)\mu_t(y)dt \, ; \, y \in \mathcal{A}(A, A_0)\} \to 0."$$

We assume that the above statement is false. This means that there exists $x \in E^{'}(M,\tau)$, $(x_n)_n \subseteq \Omega(x)$, $x_n \to 0(\mu)$ and $\epsilon > 0$, $(y_n)_n \subseteq \mathcal{A}(A, A_0)$ such that

$$\int_0^\infty \mu_t(x_n)\mu_t(y_n)dt > 2\epsilon, \;\; n = 1, 2, ... \tag{2.3}$$

Let $y_n = \frac{ReF(B_n)}{m(B_n)}$, $n \in \mathbb{N}$. Arguing as in [M] it is not difficult to see that there exists $(y_n^1)_n \subseteq E_1$ with $\tau(y_n a) = \tau(y_n^1 a)$ for every $a \in E_1^{'}(= \Psi_1(E^{'}))$, $n \in \mathbb{N}$, and

$$\mu(y_n^1) = \mu(\frac{\Psi_1 \circ \Phi_1 \circ ReF(B_n)}{m(B_n)}),\ n \in \mathbb{N}.$$

Using the same compactness result from [DDP2] (see the above "assertion") together with (2.2) we obtain a subnet $(y^1_{k_n^1})_n \subseteq (y_n^1)_n$ which converges weakly on the "direction" of $|T_1|$ (i.e. $(\tau(y^1_{k_n^1} a))_n$ converges for every $a \in E_1^{'}$). This implies that $(y_{k_n^1})_n$ converges weakly on the "direction" of $|T_1|$. In the same way we find a subnet $(y_{k_n^2})_n \subseteq (y_{k_n^1})_n$ which converges weakly on the "directions" of $|T_1|$ and $|T_2|$ and if we take the diagonal subnet we get in fact a subnet of $(y_n)_n$ which converges weakly on each "direction" $|T_1|, |T_2|, ...$ So, we can assume without loss of generality that $(y_n)_n$ itself has this property.

By the density of $\mathcal{C}$ in the norm of $E(M,\tau)$ and (2.3), we can select a net $(T_{k_n})_n \subseteq \mathcal{C}$ with the properties

$$\int_0^\infty \mu_t(x_n)\mu_t(T_{k_n})dt > 2\epsilon,\ n = 1,2,..., \tag{2.4}$$

and $\|y_n - T_{k_n}\|_{E(M,\tau)} \to 0$. Let us put $V_n := T_{k_n},\ n \in \mathbb{N}$. Since (M,τ) is diffuse it follows from (2.4) and [FK] that there exists $z_n \in E^{'}_{k_n}(:= \Psi_{k_n}(E^{'}))$ with $\mu(z_n) = \mu(x_n)$ and

$$|\tau(z_n V_n)| > 2\epsilon,\ n = 1,2,.... \tag{2.5}$$

In general, the map $f : (\Omega(x), \mu) \to \mathbb{C},\ f(z) = \tau(zy)$ is continuous for a fixed $y \in E(M,\tau)$ (cf. [DDP2]). But we know that $z_n \to 0(\mu)$ and that's why we can assume (tacking a subnet if necessary) that

$$|\tau(z_{n+1}V_n)| < \epsilon,\ n = 1,2,.... \tag{2.6}$$

Since $\|y_n - V_n\|_{E(M,\tau)} \to 0$ it follows that $(V_n)_n$ also converges on each "direction" of $|T_1|, |T_2|, ...$ This means that if we let $w_n = V_n - V_{n-1}$, $n \in \mathbb{N}$, we have $\tau(w_n a) \to 0$ for every $a \in \bigcup E_n^{'}$. Using (2.5) and (2.6) we obtain

$$|\tau(z_n w_n) > \epsilon,\ n = 1,2,.... \tag{2.7}$$

Let $Z = \{z_n; n \in \mathbb{N}\}\bigcup\{0\}$. The space $(\Omega(x), \mu)$ is a complete metric space (cf. [DDP2]) and so, (Z,μ) is a complete metric subspace of $(\Omega(x),\mu)$. So, if we define now the functions

$$f_n : (Z,\mu) \to \mathbb{C},$$

$$f_n(z) = \tau(zw_n),\ z \in Z,\ n = 1,2,...,$$

we deduce that $(f_n)_n$ are continuous functions and $f_n(z) \to 0$ for every $z \in Z$. Using Baire's theorem we obtain $N \in \mathcal{V}(0)$ in (Z,μ) and $n_0 \in \mathbb{N}$ with the property $|\tau(zw_n)| \leq \frac{\epsilon}{2}$ for every $z \in N$ and $n \geq n_0$. But this implies that there exists $m_0 \in \mathbb{N}$ such that

$$|\tau(z_m w_n)| \leq \frac{\epsilon}{2},\ \ m \geq m_0,\ n \geq N_0 \tag{2.8}$$

which contradicts (2.7). The case $\tau(1) < \infty$ is proved. The general situation when $\tau(1) = \infty$ can be "reduced" to the first one by standard arguments. We give a sketch of proof. Let again $F \in V_\Lambda^\infty(E(M,\tau))$. First, using lemma 2.1 (a) we remark (cf.

for instance [X2]) that we can assume without loss of generality that there exists a countable set of mutual disjoint projections $(P_n)_n \subseteq M$, $\tau(P_n) < \infty$, $n \in \mathbb{N}$, with $\sum_n P_n = 1$. We put $Q_n = P_1 + ... + P_n$, $n \in \mathbb{N}$. By the finite case, there exists $\phi_n \in L^\infty(E(Q_n M Q_n, \tau))$, $n \in \mathbb{N}$, with $Q_n F Q_n = \phi_n \cdot dm$, $n \in \mathbb{N}$. E has a.c. norm (cf. [LT]); thus it is not difficult to see that $Q_n F(A) Q_n \to F(A)$ in $E(M,\tau)$ for any $A \subseteq T$ and also $(\phi_n(t))_n$ is Cauchy in $E(M,\tau)$ a.e. $t \in T$. Let $\phi(t) = \lim_n \phi_n(t)$ a.e. $t \in T$. Since for every $A \subseteq T$ there exists $\lim_n \int_A \phi_n(t) dm(t) (= \lim_n Q_n F(A) Q_n = F(A))$ we deduce that ϕ is integrable (cf. [DU]) and moreover $F = \phi \cdot dm$ which completes the proof. □

For $(M,\tau) = (B(H), tr)$ we get $E(M,\tau) = C_E$, the usual Schatten class, and we obtain the following consequences.

Corollary 2.3. *We consider $\Lambda \subseteq \mathbb{Z}$. Then, E has Λ-RNP if and only if C_E has Λ-RNP.*

Corollary 2.4 ([X1])**.** *E has ARNP if and only if its non-commutative analogue $E(M,\tau)$ has ARNP.*

Corollary 2.5 ([X2])**.** *E has RNP if and only if its non-commutative analogue $E(M,\tau)$ has RNP.*

In [BD] it is proved that any Banach lattice not containing c_0 has ARNP. The following corollary can be considered as a non-commutative version of this result.

Corollary 2.6. *A non-commutative symmetric space $E(M,\tau)$ has ARNP if and only if it does not contain c_0.*

Proof. The first "implication" is obvious since c_0 does not have ARNP. For the converse, let us observe that E does not contain c_0 as well since it is isometric isomorphic with a subspace of $E(M,\tau)$. So, E has ARNP (cf. [BD]) and then, by corollary 2.4 $E(M,\tau)$ has ARNP. □

3. Remarks

We want to point out here that the above results hold also in a more general context. For this, we have to recall the notion of Radon Nikodym Property associated with subsets of countable discrete abelian groups, introduced in [E] and [D]. Let G be a compact abelian metrizable group, Γ the dual group of G, $\mathcal{B}(G)$ the σ-algebra of Borel sets of G and λ the normalised Haar measure on G. We consider also B a Banach space. If μ is a B-valued measure on $\mathcal{B}(G)$ and $\gamma \in \Gamma$ then the Fourier coefficient $\tilde{\mu}(\gamma)$ is defined by

$$\tilde{\mu}(\gamma) = \int_G \overline{\gamma(x)} d\mu(x).$$

If $\Lambda \subseteq \Gamma$ we denote as in [D] by $V^\infty_\Lambda(G,B)$ (resp. $V^1_{\Lambda,\mathrm{ac}}(G,B)$) the set of vector measure with bounded L^∞-variation such that $\tilde{\mu}(\gamma) = 0$ for all $\gamma \in \Gamma \setminus \Lambda$ (resp. the set of vector measure with bounded variation, a.c. with λ and having the same property). Following [E] and [D] we recall that B is said to have I-Λ-RNP (resp. II-Λ-RNP) if and only if every measure from $V^\infty_\Lambda(G,B)$ (resp. $V^1_{\Lambda,\mathrm{ac}}$) is representable. With the same arguments as above, the following theorem can be proven.

Theorem 3.1. *With the above notations, a rearrangement invariant space E has I-Λ-RNP* (*resp. II-Λ-RNP*) *if and only if its non-commutative analogue $E(M,\tau)$ has I-Λ-RNP* (*resp. II-Λ-RNP*).

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