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KRULL DIMENSION OF MODULES OVER INVOLUTION RINGS. II

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Abstract. Let R be a ring with involution and invertible 2, and let $\bar{S}$ be the subring of R generated by the symmetric elements in R. The following questions of Lanski are answered positively:
(i) Must $\bar{S}$ have Krull dimension when R does?
(ii) Is every Artinian R-module Artinian as an $\bar{S}$-module?

Throughout this paper R is a ring with involution $*$ in which 2 is invertible. We shall denote by $S(R)$ the set of symmetric elements of R and by $\bar{S}(R)$ the subring of R generated by $S(R)$.

All modules in this paper are assumed to be unital left modules. Given a ring A and an A-module M, the Krull dimension of M will be denoted by $k_R(M)$.

Chain conditions in rings with involution were studied in a number of papers (see, for example, [2], [4], [8], [9], [10], [11]). In [10] Lanski conjectured that if R has Krull dimension, then $\bar{S}(R)$ also must have Krull dimension. He raised implicitly a more general question, namely whether every R-module with Krull dimension has the same Krull dimension as an $\bar{S}(R)$-module. He frankly admitted that he did not know the answer even in the case of Artinian modules. In [2] the general question was affirmatively answered in the case when R is Noetherian with respect to two-sided $*$-ideals. The main aim of this paper is to prove the following:

A. *Every Artinian R-module is Artinian as an $\bar{S}(R)$-module.*

B. *If R is a ring with Krull dimension and M is an R-module, then $k_R(M) = k_{\bar{S}(R)}(M)$.*

The latter settles in particular Lanski's conjecture above.

The idea of the proof is similar to that in [2]. However we now apply some additional deep results and change the technique of the proof in some crucial points.

Note that if elements of $S(R)$ commute, then $S(R) = \bar{S}(R)$. In [1] Amitsur proved that if $\bar{S}(R)$ is a PI ring (e.g. if it is commutative), then so is R.

For every $*$-ideal I of R, R/I is a ring with involution in a natural way and $S(R/I) = (S(R) + I)/I$. Indeed, if $r + I \in S(R/I)$, then $r^* + I = r + I$. Hence $r + I = (r + r^*)/2 + I \in S(R) + I$. The other inclusion is clear.

An R-module M will be called $*$-*faithful* if for every non-zero $*$-ideal I of R, $IM \neq 0$.

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Note that if M is an R-module and T is the sum of all $*$-ideals of R annihilating M, then M is a $*$-faithful (R/T)-module. Obviously the Krull dimensions of M over R and over R/T coincide. We shall need the following

Proposition 1. *If R is a PI ring having a $*$-faithful module M with Krull dimension, then the prime radical $\beta(R)$ of R is nilpotent.*

Proof. In [12] Markov proved that if a PI ring A has a faithful module with Krull dimension, then $\beta(A)$ is nilpotent. Let $a(M) = \{r \in R \mid rM = 0\}$. Obviously $a(M)$ is an ideal of R and M is a faithful $R/a(M)$-module with Krull dimension. Hence by Markov's theorem, $\beta(R/a(M))$ is nilpotent. Clearly $(\beta(R) + a(M))/a(M) \subseteq \beta(R/a(M))$. This implies that there exists an n such that $(\beta(R))^n \subseteq a(M)$, so $(\beta(R))^n M = 0$. However $(\beta(R))^n$ is a $*$-ideal of R and M is a $*$-faithful R-module, so $(\beta(R))^n = 0$. □

We shall also need the following general results which were proved in [2].

Proposition 2. *Let S be a subring of a ring R and let I be an ideal of R such that $I \subseteq S$. Suppose that every (R/I)-module has the same Krull dimension when considered as an (R/I)-module and as an (S/I)-module. Then every R-module has the same Krull dimension as an R-module and as an S-module.*

Proposition 3. *Let S be a subring of a ring A and I a nilpotent ideal of A. Then $k_{(S+I)}(M) = k_S(M)$, for every $(S+I)$-module M.*

§1. Some reductions

In this section we make some auxiliary reductions.

Proposition 4. *Suppose that there exists an R-module M such that for some ordinal α, $k_R(M) = \alpha$ but $k_{\bar{S}(R)}(M) \neq \alpha$. Then we can assume that $\bar{S}(R) = S(R)$ is a central subring of R and R is a semiprime PI-ring.*

Proof. By [6] (Proof of Lemma 1.3), the ideal I of R generated by $\{xy - yx \mid x, y \in \bar{S}(R)\}$ is contained in $\bar{S}(R)$. Clearly, I is a $*$-ideal of R, so $S(R/I) = (S(R)+I)/I = \bar{S}(R)/I$. Applying Proposition 2 we can pass to the factor ring R/I and hence we can assume that $S(R) = \bar{S}(R)$ is commutative. From the quoted Amitsur's result it follows that R is a PI ring. Factoring out the sum T of all $*$-ideals of R annihilating M, we can assume that M is $*$-faithful. Now applying Proposition 1 we get that $\beta(R)$ is nilpotent. Hence Proposition 3 implies that $k_{(S(R)+\beta(R))}(M) \neq \alpha$. Now $S(R/\beta(R)) = (S(R) + \beta(R))/\beta(R)$, so applying Proposition 2 we can factor out $\beta(R)$ and assume that R is semiprime. In [10], page 406, Lanski proved that if $S(R)$ is commutative, then for every $s \in S(R)$, the ideal of R generated by $\{sr - rs \mid r \in R\}$ is nilpotent. This and semiprimeness of R imply that $S(R)$ is central. □

Proposition 5. *Suppose that there exists an Artinian R-module which is not Artinian as an $\bar{S}(R)$-module. Then we can assume that R is semiprime and $S(R) = \bar{S}(R)$ is a central subring of R such that every $0 \neq s \in S(R)$ is regular in R and satisfies $sM = M$.*

Proof. By Proposition 4 we can assume that $S(R) = \bar{S}(R)$ is a central subring of R. Obviously we can assume that all proper submodules of M are Artinian as $S(R)$-modules. These properties do not change when passing to the factor ring

R/T, where T is the sum of all $*$-ideals of R annihilating M. Thus we can assume that M is a $*$-faithful R-module. By Proposition 1, $\beta(R)$ is nilpotent.

Since M is a $*$-faithful R-module and $S(R)$ is a central subring of R, for every $s \in S$, $a(s) = \{m \in M \mid sm = 0\}$ is a proper R-submodule of M. Consequently $a(s)$ is an Artinian $S(R)$-module. The R-modules $M/a(s)$ and sM are isomorphic, so sM cannot be a proper submodule of M. Thus $sM = M$.

Now let $I = \{r \in R \mid rs = 0\}$. Clearly, I is a $*$-ideal of R. Moreover $IM = I(sM) = (Is)M = 0$. Since M is a $*$-faithful R-module, $I = 0$. This shows that s is regular in R.

Since $\beta(R)$ is nilpotent, $\beta(R)M \neq M$. Hence $\beta(R)M$, being a proper R-submodule of M, is Artinian as an $S(R)$-module. Consequently $\bar{M} = M/\beta(R)M$ is an Artinian $R/\beta(R)$-module, which is not Artinian as an $(S(R)+\beta(R))/\beta(R) = S(R/\beta(R))$-module. Clearly, for every $0 \neq \bar{s} = s + \beta(R) \in S(R/\beta(R))$, we have $\bar{s}\bar{M} = \bar{M}$. Moreover $I = \{r \in R \mid rs \in \beta(R)\}$ is a $*$-ideal of R such that $I^n s^n = (Is)^n = 0$ for some n. Since s is regular in R, we have $I^n = 0$. This shows that $\bar{s}$ is regular in $R/\beta(R)$. Consequently we can replace R by $R/\beta(R)$ obtaining a ring with the desired properties. □

The following was proved in [2], Proposition 1.

Proposition 6. *Let S be a subring of a ring A. Suppose there exist an ordinal $\alpha \neq 0$ and an A-module M such that $k_A(M) = \alpha$ but $k_S(M) \neq \alpha$, and suppose further that α is the least ordinal with this property. Then the module M contains an R-submodule K such that $k_S(K) \not\leq \alpha$ and, for every R-submodule N of K, either $k_S(N) \leq \alpha$ or $k_S(K/N) < \alpha$.*

Now we shall prove

Proposition 7. *Suppose that R has Krull dimension and that there exist an ordinal α and R-module M with $k_R(M) = \alpha$ and $k_{\bar{S}(R)}(M) \neq \alpha$. We can assume that*

(i) *R is a $*$-prime ring and $S(R) = \bar{S}(R)$ is a central subring of R;*
(ii) *for every $0 \neq c \in S(R)$, $k_{S(R)}(M/cM) < \alpha$.*

Proof. Suppose that we have chosen R with the least possible Krull dimension γ and then M an R-module with the least possible Krull dimension α. Observe that $\gamma > 0$. Indeed, if $\gamma = 0$, then R is an Artinian ring. Hence R is a Noetherian ring and by [2], for every R-module M, $k_R(M) = k_{\bar{S}(R)}(M)$.

Note that all the reductions in Proposition 4 do not increase the Krull dimension of the ring. Thus we can apply the proposition and assume that $S(R) = \bar{S}(R)$ is a central subring of R. Moreover, since R has Krull dimension, $\beta(R)$ is nilpotent (here we could also apply arguments used in the proof of Proposition 5). Similarly as in the proof of Proposition 4, applying Propositions 3 and 2, we can factor out $\beta(R)$ and assume that R is a semiprime ring. Since R has Krull dimension, it has only a finite number of minimal prime ideals ([5], Proposition 7.3). Note that if P is a prime ideal of R, then $P \cap P^*$ is a $*$-prime ideal of R. Consequently there are $*$-prime ideals P_1,...,P_n of R such that $P_1 \cap ... \cap P_n = 0$. Let $M_i = P_i P_{i-1} ... P_1 M$, $1 \leq i \leq n$. Note that $0 = M_n \subseteq M_{n-1} \subseteq ... \subseteq M_1 \subseteq M_0 = M$ and all M_i are R-submodules of M. Hence for some $0 \leq i \leq n-1$, $k_{S(R)}(M_i/M_{i+1}) \not\leq \alpha$. Now $M_{i+1} = P_{i+1}M_i$, so $k_{S(R)}(M_i/M_{i+1}) = k_{(S(R)+P_{i+1})/P_{i+1}}(M_i/M_{i+1}) = k_{S(R/P_{i+1})}(M_i/M_{i+1})$. Obviously $k_{R/P_{i+1}}(M_i/M_{i+1}) \leq \alpha$ and $S(R/P_{i+1})$ is a central subring of R/P_{i+1}. This proves (i).

Since R is $*$-prime and $S(R)$ is central, each non-zero element $c \in S(R)$ is regular in R. Hence the Krull dimension of R/cR is strictly smaller than γ. By the choice of γ, $k_{R/cR}(M/cM) = k_{S(R/cR)}(M/cM) = k_{(S(R)+cR)/cR}(M/cM)$. Obviously $k_{R/cR}(M/cM) = k_R(M/cM) \leq \alpha$ and, since $S(R/cR) = (S(R)+cR)/cR$, $k_{R/cR}(M/cM) = k_{S(R)}(M/cM)$. Thus $k_{S(R)}(M/cM) \leq \alpha$. This implies that $k_{S(R)}(cM) \not\leq \alpha$. Hence by Proposition 6, we may assume that $k_{S(R)}(M/cM) < \alpha$. The proof is complete. ∎

§2. Main results

We shall need the following results.

Proposition 8 ([14, Theorem 7.6]). *Let σ be an automorphism of order 2 of a ring A with $1/2 \in A$ and let $A^\sigma = \{a \in A \mid \sigma(a) = a\}$ be the fixed subring of A with respect to σ. Then $k_A(M) = k_{A^\sigma}(M)$ for every A-module M.*

Proposition 9 ([7, Theorem 3.10]). *Suppose that R is non-commutative and semiprime and $S = S(R)$ is a field. Then*

(i) *S is the center of R;*
(ii) *R is a division ring with $[R : S] = 4$ or R is isomorphic to the ring $M_2(S)$ of 2×2 - matrices over S.*

Proposition 10 ([13, Theorem 1]. *Suppose that K is a field of characteristic $\neq 2$ and $*$ is a K-involution of the ring $M_2(K)$. Then, in an appropriate K-basis, $*$ is one of the following types:*

(transpose involution) $\begin{pmatrix} a & b \\ c & d \end{pmatrix}^* = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$

or

(symplectic involution) $\begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} d & -b \\ -c & a \end{pmatrix}$

for all a, b, c, d in K.

In the former case, $S(M_2(K))$ is not a subring of $M_2(K)$.

Now we can prove

Theorem 1. *Suppose that M is an R-module. Suppose further that*

(i) *R is semiprime, $S = S(R)$ is a central subring of R and every non-zero element of S is regular in R;*
(ii) *$k_R(M) = \alpha$ and for every $0 \neq s \in S$, $k_S(M/sM) < \alpha$.*

Then $k_S(M) = \alpha$.

Proof. Suppose the theorem is not true and α is the least possible ordinal for which it does not hold. Obviously $\alpha > -1$. By Proposition 6 we can assume that M is an R-module such that $k_R(M) = \alpha$, $k_S(M) \neq \alpha$ and for every R-submodule N of M, $k_S(N) \leq \alpha$ or $k_S(M/N) < \alpha$.

We claim that for every $0 \neq c \in S$, $k_{S+cR}(M) \leq \alpha$. If not, then there is a strictly descending chain $\{M_i \mid i = 0, 1, 2, ...\}$ of $(S+cR)$-submodules of M such that $M_0 = M$ and for every $i \geq 1$, $k_{(S+cR)}(M_{i-1}/M_i) \not< \alpha$. This in particular implies that for $N = RM_1$, $k_S(N) \not\leq \alpha$. Since N is an R-submodule of M, the assumption on M we made at the beginning of the proof gives that $k_S(M/N) < \alpha$. Consequently $k_S(cM/cN) < \alpha$. By (ii), $k_S(M/cM) < \alpha$, so $k_S(M/cN) < \alpha$. However $cN = cRM_1 \subseteq M_1$, so $k_S(M_0/M_1) = k_S(M/M_1) \leq k_S(M/cM) < \alpha$. This contradiction proves the claim.

If R is commutative, then $*$ is an automorphism of R of order 2 the fixed ring of which is equal S. We get a contradiction with Proposition 8. Thus R must be non-commutative. We can form the ring $Q = S^{-1}R$ of quotients of R with respect to the non-zero elements of S. The involution can be extended uniquely to Q and then $K = S(Q)$ is the field of fractions of the domain S. Now Propositions 9 and 10 allow us to assume that Q is the ring of 2×2 - matrices over K with the symplectic involution or that Q is a division ring with center K and $[Q : K] = 4$.

We claim that there exists a skew-symmetric element $x \in R$ such that $x^2 \neq 0$ and $B = \{z \in Q \mid zx = xz\}$ is equal to the K-subalgebra $K[x]$ of Q generated by x. Suppose first that Q is a division ring. Since 2 is invertible in R, for every $0 \neq r \in R \setminus S$, $x = r - r^*$ is a skew-symmetric element of R not belonging to K. Obviously $x^2 \neq 0$. Since $x^2 \in K$ and $x \notin K$, $\dim_K(K[x]) = 2$. Therefore $K[x]$ is a maximal subfield of Q and $K[x] = B$. Suppose now that Q is the ring of 2×2 - matrices over K with the symplectic involution. Then Q contains a set $E = \{e_{ij} \mid i, j = 1, 2\}$ of matrix units such that $e_{12}^* = -e_{12}$ and $e_{21}^* = -e_{21}$. Clearly $y = e_{12} + e_{21}$ is a skew-symmetric invertible element of Q. There exists $0 \neq s \in S$ such that $x = sy \in R$. Obviously x is a skew-symmetric element of R and $x^2 \neq 0$. Direct computations show that $B = K[x]$. The claim is proved.

Let σ be the inner automorphism of Q induced by x. Since $x^2 \in S$ is a central element of R (and so of Q), $\sigma^2 = id_Q$. We also note that σ commutes with the involution. Observe that for $s = x^2$, $A = S + sR + xRx$ is a σ-invariant and $*$-invariant subring of R. We already know that $k_{S+sR}(M) \leq \alpha$. Since the lattice of A-submodules of M is a sublattice of the lattice of $(S + sR)$-submodules of M, we conclude that $k_A(M) \leq \alpha$. By Proposition 8 we have that $k_A(M) = k_{A^\sigma}(M)$. Clearly $A^\sigma \subseteq B = \{z \in Q \mid zx = xz\}$ and so A^σ is a commutative ring. Recalling that σ commutes with $*$, we conclude that A^σ is a $*$-invariant subring of R containing S. Therefore $*$ induces an automorphism of order two on A^σ. Noting that $S(A^\sigma) = S$ and again applying Proposition 8, we infer that

$$k_S(M) = k_{A^\sigma}(M) = k_A(M) \leq \alpha.$$

Since the lattice of R-submodules of M is a sublattice of the lattice of S-submodules of M, we conclude that $k_R(M) \leq k_S(M)$. Thus $k_S(M) = \alpha$ which contradicts our assumption. The proof is complete. □

Theorem 1 and Proposition 5 give

Corollary 1. *Every Artinian R-module is Artinian as an $\bar{S}(R)$-module.*

Applying Theorem 1 and Proposition 7 we get

Corollary 2. *If R is a ring with Krull dimension and M is an R-module, then $k_R(M) = k_{\bar{S}(R)}(M)$.*

We close with a result which sheds some further light on Lanski's general problem. Note that in particular it shows that the general question can be reduced to the study of $*$-faithful modules over semiprime Goldie rings.

Theorem 2. *Suppose that R contains an invertible skew-symmetric element x. Then for every R-module M, $k_R(M) = k_{\bar{S}(R)}(M)$.*

Proof. Suppose that there exists an R-module M with $k_R(M) \neq k_{\bar{S}(R)}(M)$. Applying Proposition 4 we can assume that R is a semiprime PI ring and $S = \bar{S}(R)$ is a central subring of R. Factoring out the sum of all $*$-ideals of R annihilating

M, we can assume that M is $*$-faithful. Now by Proposition 1 we get that $\beta(R)$ is nilpotent. Set $I = \beta(R)$, $I^0 = R$, $M_t = I^t M / I^{t+1} M$ and $N = \bigoplus_{t=0}^{n} M_t$, where $I^{n+1} = 0$. We claim that N is a $*$-faithful R/I-module. Indeed, if J is a $*$-ideal of R such that $(J/I)N = 0$, then $JI^t M \subseteq I^{t+1} M$ for $t = 0, 1, ..., n$. It follows that $J^{n+1} M \subseteq I^{n+1} M = 0$. Hence, since M is $*$-faithful, $J^{n+1} = 0$. Therefore $J = I$, which proves the claim. Now passing to the ring R/I and the module N and applying Propositions 2 and 3 we can assume that R is semiprime and M is $*$-faithful. Let $A = \{r \in R \mid rM = 0\}$ be the annihilator of M. Clearly A is an ideal of R and M is a faithful R/A-module. Hence by [12], the prime radical B/A of R/A is nilpotent, i.e., $B^{m+1} \subseteq A$ for a non-negative integer m. Similarly as above one checks that $N = \bigoplus_{t=0}^{m} B^t M / B^{t+1} M$ is a faithful R/B-module. It is known [3] that every semiprime PI ring having a faithful module with Krull dimension is a (left and right) Goldie ring. Hence R/B is a Goldie ring. Note that since M is a $*$-faithful R-module, $A \cap A^* = 0$. Moreover $(B \cap B^*)^{m+1} \subseteq A \cap A^* = 0$. Hence since R is semiprime, $B \cap B^* = 0$. Now R/B^* is antiisomorphic to R/B, so R/B^* is a Goldie ring. Since B is isomorphic to the ideal $(B + B^*)/B^*$ of R/B^*, B is also a Goldie ring. Let $I = \{r \in R \mid rB = 0\}$ be the annihilator of B in R. Since R is semiprime, $I \cap B = 0$. Hence I is isomorphic to the ideal $(I + B)/B$ of R/B, so I is a Goldie ring. Finally $I + B = I \oplus B$ is a Goldie ring which is an essential ideal of R. Hence R is a Goldie ring. Let T be the set of all regular elements in S. Note that if c is a regular central element of R, then c^* and cc^* are regular central elements of R and $cc^* \in T$. Hence regular central elements of R are invertible in the classical ring of quotients $Q = T^{-1}R$. Now by Proposition 5.8 in [15], Q is a semisimple Artinian ring, so $Q = \bigoplus_{i=1}^{n} A_i$, where A_i are simple rings. We can extend the involution from R to Q putting $(r/t)^* = r^*/t$, for $r \in R$, $t \in T$. Then $S(Q) = T^{-1}S$ and hence $S(Q)$ is a central subring of Q. Obviously for every $1 \leq i \leq n$ there is $1 \leq j \leq n$ such that $A_i^* = A_j$. If $i \neq j$, then commutativity of $S(Q)$ implies that both A_i and A_j are fields. If $i = j$, then by Propositions 9 and 10, A_i is isomorphic to the ring of 2×2-matrices over a field with the symplectic involution or it is a division ring of dimension 4 over its center. In both cases $S(A_i)$ coincides with the center of A_i.

Denote by σ the inner automorphism of R induced by x. Since $x^2 \in S$ is a central element, $\sigma^2 = id_R$. We claim that R^σ is a commutative subring of R. Indeed, it is enough to show that Q^σ is a commutative subring of Q. The foregoing show that it suffices to prove the claim in the case when Q is the ring of 2×2-matrices over a field with the symplectic involution or Q is a division ring of dimension 4 over its center. We can reduce the latter case to the former by tensoring Q by the algebraic closure of its center. Thus we can assume that $Q = M_2(F)$, where F is an algebraically closed field and $S(Q) = F$. Since $x^2 \in F$, we conclude that the Jordan normal form of x is $\left(\begin{smallmatrix} c & 0 \\ 0 & -c \end{smallmatrix}\right)$ for some $0 \neq c \in F$. It follows that Q^σ is isomorphic to a subring of the ring $\left(\begin{smallmatrix} F & 0 \\ 0 & F \end{smallmatrix}\right)$, which is obviously commutative. The claim is proved. Note now that R^σ is a $*$-invariant subring of R and $S = \{r \in R^\sigma \mid r^* = r\}$. Clearly $*$ induces on R^σ an automorphism of order two. Applying twice Proposition 8 we get that $k_R(M) = k_S(M)$, a contradiction. The proof is complete. □

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