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CONGRUENCES ON "CHARACTER" VALUES OF PERMUTATION SUMMANDS

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Abstract. A class of congruences on "character" values Φ_L of a permutation summand L are exhibited, from which follows the connectedness of the prime ideal spectrum of the Grothendieck ring of permutation summands.

Let G be a finite group and A the ring of integers in a number field K. An AG-lattice is called a *permutation lattice* if it has an A-basis, necessarily finite, which is permuted by the action of G. It will be called a *permutation summand* (for G over A), if it is a direct summand, as AG-module, of a permutation lattice. The Grothendieck ring $\Omega_A(G)$ of the category of all permutation summands for G over A has been studied in [3], via a sort of numerical character Φ_L of a permutation summand L. The construction of Φ_L is reviewed in the first paragraph of the proof below.

In this note we exhibit a class of congruences on the values of Φ_L which are strong enough to imply the connectedness of the prime ideal spectrum of $\Omega_A(G)$. The corresponding result for the character ring $R_K(G)$ was established in [2], where Lemma 7 gives analogous congruences on character values. For the Burnside ring $\Omega(G)$ of finite G-sets, the connectedness fails [1], because there are too few congruences on the number of fixed points of G-sets.

The function Φ_L takes values in the ring A' of integers of some sufficiently large number field, for instance $K(\zeta_{|G|})$, and is defined on triples $(H, b, \mathfrak{p}')$ of G over A. Here $\mathfrak{p}'$ is a non-zero prime ideal of A' so that if p is the unique prime number in $\mathfrak{p}'$ then H is a p-hypoelementary subgroup of G and b is a generator of $H/O_p(H)$ where $O_p(H)$ is the largest normal p-subgroup of H.

Congruences. *For any prime number q, we have*

$$\Phi_L(H, b, \mathfrak{p}') \equiv \Phi_L(O^q(H), b_{q'}, \mathfrak{p}') \ mod\ \mathfrak{q}'$$

where $O^q(H)$ is the smallest normal subgroup of H with $H/O^q(H)$ a q-group, $b_{q'}$ is the q'-part of the element b, and $\mathfrak{q}'$ is any prime ideal above q.

Proof. Notations are consistent with those used in [3]. Let $i_{\mathfrak{p}'} : A' \to A'_{\mathfrak{p}'}$ be the inclusion of A' in its completion at $\mathfrak{p}'$, and let $\mathfrak{p} = \mathfrak{p}' \cap A$. Denote the $A_{\mathfrak{p}}G$-module $A_{\mathfrak{p}} \otimes_A L$ by M for simplicity. Since H is p-hypoelementary, $O_p(H)$ is the normal p-Sylow subgroup of H. Decompose the restriction M_H of M to H as $M_H \simeq M' \oplus M''$, where the vertices of the indecomposable $A_{\mathfrak{p}}H$-summands of M' are $O_p(H)$, and

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the vertices of M'' are proper subgroups of $O_p(H)$. By the definition of Φ_L (cf. [3] (2.1)), we have

$$i_{\mathfrak{p}'}\Phi_L(H, b, \mathfrak{p}') = \text{ trace of } b \text{ acting on } M'.$$

If the action of b on M' has eigenvalues $\lambda_1, ..., \lambda_r$ in $A'_{\mathfrak{p}'}$, then $\Phi_L(H, b, \mathfrak{p}') = \sum_i \xi_i$, where ξ_i is the preimage of λ_i under $i_{\mathfrak{p}'}$. We will call this the **pretrace** of b on M' for convenience.

Denote $O_p(O^q(H)) = O_p(H) \cap O^q(H)$ by Q, and further decompose M'' as $M'' \simeq M_1'' \oplus M_2''$, where the vertices of the indecomposable $A_{\mathfrak{p}}H$-summands of M_1'' contain Q and those of M_2'' do not.

Since every indecomposable $A_{\mathfrak{p}}H$-summand of $M' \oplus M_1''$ has vertex P between Q and $O_p(H)$ from the above decomposition of M_H, it is an $A_{\mathfrak{p}}H$-summand of $\mathrm{ind}_P^H(A_{\mathfrak{p}})$ by [3](1.1). Its restriction to $O^q(H)$ is then an $A_{\mathfrak{p}}O^q(H)$-summand of $\mathrm{ind}_Q^{O^q(H)}(A_{\mathfrak{p}})$ by Mackey decompositon, hence has vertex Q. Every indecomposable summand of the restriction $(M_2'')_{O^q(H)}$ has vertex properly contained in Q as the vertex can only drop after restriction. Therefore from the above decomposition of M_H, the restriction of M to $O^q(H)$ has the decomposition $M_{O^q(H)} \simeq (M' \oplus M_1'')_{O^q(H)} \oplus (M_2'')_{O^q(H)}$, where the vertices of the indecomposable $A_{\mathfrak{p}}O^q(H)$-summands of $(M' \oplus M_1'')_{O^q(H)}$ are Q, and the vertices of $(M_2'')_{O^q(H)}$ are proper subgroups of Q. Again by the definition of Φ_L, applied to $(O^q(H), b_{q'}, \mathfrak{p}')$, we obtain

$$\Phi_L(O^q(H), b_{q'}, \mathfrak{p}') = \text{ pretrace of } b_{q'} \text{ acting on } M' \oplus M_1''.$$

Now the congruence follows from

Claim. i) pretrace of b on $M' \equiv$ pretrace of $b_{q'}$ on M' mod $\mathfrak{q}'$;
ii) pretrace of $b_{q'}$ on $M_1'' \equiv 0$ mod $\mathfrak{q}'$.

Proof of Claim. i) If m is a sufficiently large power of q, we have $b^m = b_{q'}^m$, and the eigenvalues of b^m on M' have preimages $\xi_1^m, ..., \xi_r^m$ under $i_{\mathfrak{p}'}$. Hence

$$(\text{pretrace of } b)^m = (\sum_i \xi_i)^m \equiv \sum_i \xi_i^m = \text{pretrace of } b^m \text{ mod } \mathfrak{q}'$$

and, for the same reason,

$$(\text{pretrace of } b_{q'})^m \equiv \text{pretrace of } b_{q'}^m \text{ mod } \mathfrak{q}'.$$

Combining gives

$$(\text{pretrace of } b)^m \equiv (\text{pretrace of } b_{q'})^m \text{ mod } \mathfrak{q}'$$

from which i) follows.

ii) We may assume $M_1'' \neq 0$. Then $Q \subsetneq O_p(H)$, hence p must be equal to q, and $H/Q = (O_p(H)/Q) \times (O^q(H)/Q)$ is nilpotent.

Since Q acts trivially on M_1'' by [3](1.1), M_1'' can be considered as an $A_{\mathfrak{p}}H/Q$-module. By [4]§2, this module has the structure,

$$M_1'' \simeq \sum_j N_j \otimes_{A_{\mathfrak{p}}} \mathrm{ind}_{D_j}^{O_p(H)/Q}(A_{\mathfrak{p}})$$

for some $A_{\mathfrak{p}}O^q(H)/Q$-lattices N_j and some p-subgroups D_j of $O_p(H)/Q$. These D_j are actually the vertices of M_1'', hence are *properly* contained in $O_p(H)/Q$.

If the eigenvalues of $b_{q'}$ on N_j have preimages $\xi_1^{(j)}, ..., \xi_{r_j}^{(j)}$ under $i_{\mathfrak{p}'}$, then the eigenvalues of $b_{q'}$ on $N_j \otimes_{A_\mathfrak{p}} \mathrm{ind}_{D_j}^{O_p(H)/Q}(A_\mathfrak{p})$ have preimages $\xi_1^{(j)}, ..., \xi_{r_j}^{(j)}$ each repeated $|O_p(H)/Q\ : D_j|$ times. Thus

$$\text{pretrace of } b_{q'} \text{ on } M_1'' = \sum_j |O_p(H)/Q : D_j| \sum_i \xi_i^{(j)} \equiv 0 \mod pA',$$

as required. This completes the proof of the claim, hence of the congruence. □

We want to examine the prime ideal spectrum $\mathrm{Spec}(\Omega_A(G))$ of the commutative ring $\Omega_A(G)$. Let $T_G(A)$ be the set of triples $(H, b, \mathfrak{p}')$, and $(A')^{T_G(A)}$ the ring of all maps on triples with values in A'. Since the ring homomorphism $\Phi : \Omega_A(G) \to \mho_A(G)$ has a nilpotent kernel [3], and $\mho_A(G)$ is a subring of $(A')^{T_G(A)}$ with finite **Z**-rank, it induces the surjection

$$\mathrm{Spec}((A')^{T_G(A)}) \xrightarrow{\text{going-down}} \mathrm{Spec}(\mathrm{im}\Phi) \xrightarrow{\Phi^{-1}} \mathrm{Spec}(\Omega_A(G)).$$

$\mathrm{Spec}(A')$ consists of the ideal 0 and the maximal ideals of A'. The spectrum of $(A')^{T_G(A)}$ can be identified with $T_G(A) \times \mathrm{Spec}(A')$: with each $T \in T_G(A)$ and each $\mathfrak{q}' \in \mathrm{Spec}(A')$ we associate the prime ideal $\mathfrak{q}'_T$ consisting of those $f \in (A')^{T_G(A)}$ such that $f(T) \in \mathfrak{q}'$. The image of $\mathfrak{q}'_T$ in $\mathrm{Spec}(\Omega_A(G))$ is the prime ideal $P_{\mathfrak{q}',T}$ corresponding to the prime ideal $\mathfrak{q}'_T \cap \mathrm{im}\Phi$ in $\mathrm{im}\Phi$ by Φ^{-1}, i.e.

$$P_{\mathfrak{q}',T} = \{x \in \Omega_A(G) : \Phi_x(T) \in \mathfrak{q}'\}.$$

Lemma. *With above notation, then*

(1) $P_{0,T} \subset P_{\mathfrak{q}',T}$;
(2) *If $\mathfrak{q}'$ is a maximal ideal of A' above a prime number q and $T = (H, b, \mathfrak{p}')$ is a triple, we denote the triple $(O^q(H), b_{q'}, \mathfrak{p}')$ by T^q. Then $P_{\mathfrak{q}',T} = P_{\mathfrak{q}',T^q}$.*

Proof. (1) is clear.

(2) By the congruences we have $\Phi_x(T) \equiv \Phi_x(T^q) \mod \mathfrak{q}'$ for $x \in \Omega_A(G)$. Thus $x \in P_{\mathfrak{q}',T} \iff \Phi_x(T) \in \mathfrak{q}' \iff \Phi_x(T^q) \in \mathfrak{q}' \iff x \in P_{\mathfrak{q}',T^q}$. □

Corollary. $\mathrm{Spec}(\Omega_A(G))$ *is connected.*

Proof. Let C be the connected component of the point $P_{0,(1)}$ in $\mathrm{Spec}(\Omega_A(G))$ where (1) is the cyclic triple (cf. [3] §3) of the trivial subgroup. By (1) of the Lemma, the closure $\overline{\{P_{0,T}\}}$ contains $P_{\mathfrak{q}',T}$ for all $\mathfrak{q}'$. So it suffices to show that C contains $\overline{\{P_{0,T}\}}$ for every triple T. We proceed by induction on the order of the subgroup H appearing in the triple $T = (H, b, \mathfrak{p}')$.

If H is trivial this follows by the definition of C so we suppose H is non-trivial. Choose a prime number q so $O^q(H) \subsetneq H$, and a prime ideal $\mathfrak{q}'$ of A' containing q. By (2) and (1) of the Lemma we have $P_{\mathfrak{q}',T} = P_{\mathfrak{q}',T^q}$, hence $\overline{\{P_{0,T}\}} \cap \overline{\{P_{0,T^q}\}}$ is not empty, and $\overline{\{P_{0,T}\}} \cup \overline{\{P_{0,T^q}\}}$ is connected. But $\overline{\{P_{0,T^q}\}} \subseteq C$ by the induction hypothesis, and therefore $\overline{\{P_{0,T}\}} \subseteq C$. □

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