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ON A CLASS OF SUBALGEBRAS OF $C(X)$ AND THE INTERSECTION OF THEIR FREE MAXIMAL IDEALS

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Abstract. Let X be a Tychonoff space and A a subalgebra of $C(X)$ containing $C^*(X)$. Suppose that $C_K(X)$ is the set of all functions in $C(X)$ with compact support. Kohls has shown that $C_K(X)$ is precisely the intersection of all the free ideals in $C(X)$ or in $C^*(X)$. In this paper we have proved the validity of this result for the algebra A. Gillman and Jerison have proved that for a realcompact space X, $C_K(X)$ is the intersection of all the free maximal ideals in $C(X)$. In this paper we have proved that this result does not hold for the algebra A, in general. However we have furnished a characterisation of the elements that belong to all the free maximal ideals in A. The paper terminates by showing that for any realcompact space X, there exists in some sense a minimal algebra A_m for which X becomes A_m-compact. This answers a question raised by Redlin and Watson in 1987. But it is still unsettled whether such a minimal algebra exists with respect to set inclusion.

1. Introduction

One of the fascinating problems considered in Gillman and Jerison [2] is that of characterising the intersection of all the free maximal ideals in the algebra $C(X)$ of real-valued continuous functions on a Tychonoff space X and its subalgebra $C^*(X)$ of bounded functions. Suppose $C_K(X)$ is the set of all functions in $C(X)$ which have compact support, and let $C_\infty(X)$ consist of exactly those functions f in $C(X)$ which vanish at ∞ in the sense that $\{x \in X : |f(x)| \geq \frac{1}{n}\}$ is compact for each n in $\mathbb{N}$. Kohls [3] has shown that the intersection of all the free ideals in $C(X)$ or in $C^*(X)$ is $C_K(X)$. We have established the truth of the same result for a subalgebra A of $C(X)$ that contains $C^*(X)$. Kohls [3] has further proved that the intersection of all the free maximal ideals in $C^*(X)$ is precisely the set $C_\infty(X)$. Incidentally it is shown in [2] that for a realcompact space X, $C_K(X)$ is identical to the intersection of all the free maximal ideals in $C(X)$. In this paper we show that for a subalgebra A of $C(X)$ containing $C^*(X)$, each element f belonging to the intersection of all the free maximal ideals in A is characterised by the property that $\{x \in X : |f(x)g(x)| \geq \frac{1}{n}\}$ is compact for each n in $\mathbb{N}$ and for each g in A. It is interesting to note that this result puts the two earlier results into a common setting.

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Redlin and Watson [5] introduced the notion of A-compactness of which compactness and realcompactness are particular cases. According to this terminology a compact space is C^*-compact while a realcompact space is C-compact. In view of the result of the last paragraph, note that if $A = C(X)$ or $C^*(X)$ and X is A-compact, then $C_K(X)$ is identical to the intersection of all the free maximal ideals in A. We have constructed an example which shows that such a conclusion is not true in general for an arbitrary A-compact space.

We conclude the paper by showing that given any realcompact space X, there exists in some sense a minimal algebra A_m lying between $C(X)$ and $C^*(X)$ for which X becomes A_m-compact. This gives an answer to the question raised by Redlin and Watson [5]. It has further been shown that a minimal algebra thus obtained need not be minimal with respect to set inclusion, however it still remains open whether such a minimal algebra exists with respect to set inclusion.

2. Intersection of free maximal ideals

Throughout the paper X stands for a Tychonoff space and subalgebras of $C(X)$ are supposed to contain $C^*(X)$. For any f in $C(X)$, $Z(f)$ will denote the zero-set $\{x \in X : f(x) = 0\}$. Ideals of subalgebras of $C(X)$ are assumed to be proper. An ideal I in a subalgebra A of $C(X)$ is called *fixed* if $\bigcap Z[I] \neq \emptyset$, otherwise I is said to be *free*. Each member f of $C(X)$ has a unique continuous extension $f^* : \beta X \longrightarrow \mathbb{R}^*$, where $\mathbb{R}^*$ is the one-point compactification of $\mathbb{R}$; if $f \in C^*(X)$, f^* is the same as f^β, the unique extension of f to βX. Plank [4] has shown that the family of all the maximal ideals in A is precisely the set $\{M_A^p : p \in \beta X\}$ where $M_A^p = \{f \in A : (fg)^*(p) = 0 \ \forall g \in A\}$; and M_A^p is a free ideal if and only if p belongs to $\beta X - X$.

If F is the intersection of all the free ideals in A, then it is easy to show that $C_K(X) \subset F$. On the other hand, $C_K(X) = \bigcap_{p \in \beta X - X} O^p$ (see [2], 7E) where for each p in βX, $O^p = \{f \in C(X) : cl_{\beta X} Z(f) \textit{ is a neighbourhood of } p\}$. Furthermore for each p in βX, O^p is the intersection of all the prime ideals containing it and contained in M_C^p and hence

$$C_K(X) = \bigcap_{p \in \beta X - X} \{P \cap A : O^p \subset P \subset M_C^p,\ P \textit{ is a prime ideal in } C(X)\}.$$

It is clear that for any prime ideal P in $C(X)$ appearing on the right side of the above equality, $P \cap A$ is a free prime ideal of A. Hence $F \subset C_K(X)$. Thus we have the following result.

Theorem 2.1. *$C_K(X)$ is the intersection of all the free ideals in A.*

In order to describe the intersection of all the free maximal ideals in A, let $A_\infty(X)$ denote the family of all functions f in A for which the set $A_n(fg) = \{x \in X : |f(x)g(x)| \geq \frac{1}{n}\}$ is compact for each n in $\mathbb{N}$ and each g in A. If f belongs to $A_\infty(X)$, g is in A, p belongs to $\beta X - X$ and $\epsilon > 0$, then it is easy to see in view of the continuity of $(fg)^\beta$ at p and denseness of X in βX that $|(fg)^\beta(p)| < \frac{1}{n} + \epsilon$ for each n in $\mathbb{N}$. Consequently $\beta X - X \subset Z((fg)^\beta)$ and hence $A_\infty(X) \subset \bigcap\{M_A^p : p \in \beta X - X\}$. Conversely, if f belongs to M_A^p for each p in $\beta X - X$ and g is in A, then fg belongs to $C^*(X)$ and $\beta X - X \subset Z((fg)^\beta)$. We claim that $A_n(fg)$ is compact. If not, then there exists p in $cl_{\beta X} A_n(fg) - A_n(fg)$ for which $(fg)^\beta(p) = 0$. But $|fg|^\beta(cl_{\beta X} A_n(f.g)) \subset cl_{\mathbb{R}}(|f.g|(A_n(f.g))) \subset [\frac{1}{n}, \infty)$ —a contradiction. Therefore we have the following result:

Theorem 2.2. *$A_\infty(X)$ is the intersection of all the free maximal ideals in A.*

We note that if X is realcompact and $A = C(X)$, then $A_\infty(X)$ is the family of all functions in $C(X)$ with compact support, and so Theorem 8.19 of [2] follows from our Theorem 2.2. On the other hand if $A = C(X)$, then $A_\infty(X)$ and $C_\infty(X)$ are identical and hence Lemma 3.2 of [3] is also a special case of Theorem 2.2.

3. A-compactness

Following Redlin and Watson [5], we define a maximal ideal M in A to be *real* if the quotient field A/M is isomorphic to $\mathbb{R}$, otherwise M is called *hyperreal*. X is called *A-compact* if every real maximal ideal in A is fixed. In view of this definition it follows that a compact space is C^*-compact while a realcompact space is C-compact.

As in [2], 7.9(b), one can prove the following lemma.

Lemma 3.1. *For each p in βX, M_A^p is hyperreal if and only if $M_{C^*}^p$ contains a unit of A.*

In what follows we give a useful characterisation of A-compactness.

Theorem 3.2. *A space X is A-compact if and only if for every p in $\beta X - X$, there exists an f in $C^*(X)$ such that f is a unit of A and $f^\beta(p) = 0$ (or equivalently X is A-compact if and only if for every p in $\beta X - X$, there exists a unit g of A such that $g^{-1} \in C^*(X)$ and $g^*(p) = \infty$).*

Proof. Let X be A-compact and $p \in \beta X - X$. Then M_A^p is hyperreal and hence by Lemma 3.1, $M_{C^*}^p = \{h \in C^*(X) : h^\beta(p) = 0\}$ contains a unit f of A. Clearly $f^\beta(p) = 0$. Conversely, let the given condition hold. Then for any $p \in \beta X - X$, $f^\beta(p) = 0$ for some f in $C^*(X)$ with f a unit of A. Since now $f \in M_{C^*}^p$, Lemma 3.1 implies that M_A^p is hyperreal and hence X is A-compact. □

Remark 3.3. If we take $A = C(X)$ in the above theorem, then we have the following result.

X is realcompact if and only if each point of $\beta X - X$ is contained in a zero-set in βX which misses X.

This is in fact the content of a theorem of Hewitt (see [6], page 31).

Note that if X is C-compact (respectively C^*-compact), then $C_K(X)$ is the same as the intersection of all the free maximal ideals in $C(X)$ (respectively $C^*(X)$). The following example shows that this is not true for an arbitrary A-compact space. In what follows for any subfamily $\mathcal{F}$ of $C(X)$, the subset $\mathcal{A}(\mathcal{F})$ will stand for the smallest subalgebra of $C(X)$ containing $\mathcal{F}$.

Example 3.4. Consider $A = \mathcal{A}(C^*(\mathbb{N}) \cup \{i\})$, where $i(n) = n$ for each n in $\mathbb{N}$. Since $Z(j^\beta) = \beta\mathbb{N} - \mathbb{N}$, where $j = i^{-1}$, it follows from Theorem 3.2 that $\mathbb{N}$ is A-compact. Let h in $C^*(\mathbb{N})$ be defined as $h(n) = e^{-n}$ for each n in $\mathbb{N}$. Then $h^\beta(p) = 0$ for all p in $\beta\mathbb{N} - \mathbb{N}$, consequently $(hg)^*(p) = 0$ for all p in $\beta\mathbb{N} - \mathbb{N}$ and for all g in $C^*(\mathbb{N})$. Since $lim_{n\to\infty}(n^s e^{-n}) = 0$ for each s in $\mathbb{N}$, this clearly implies $(hg)^*(p) = 0$ for all p in $\beta\mathbb{N} - \mathbb{N}$ and for all g in A. Hence h belongs to every free maximal ideal in A, yet h does not belong to $C_K(\mathbb{N})$.

4. On a question raised by Redlin and Watson

Redlin and Watson [5] raised the following question: Given a realcompact space X, does there exist in some sense a minimal algebra A_m over $\mathbb{R}$ for which X is A_m-compact? In this section we give an answer to this question. We recall the well-known fact that X is σ-compact and locally compact if and only if $\beta X - X$ is a zero-set in βX (see [6], Exercise 1B).

Consider any noncompact, σ-compact and locally compact space X. Then there exists an f in $C^*(X)$ for which $\beta X - X = Z_{\beta X}(f^\beta)$. Let $g = f^{-1}$ and $A = \mathcal{A}(C^*(X) \cup \{g\})$. Then Theorem 3.2 implies that X is A-compact. Also X is not C^*-compact. It might be tempting to conjecture that A is the smallest subalgebra of $C(X)$ with respect to the set inclusion relation for which X becomes A-compact. That this is false for a suitable choice of X is established in the following example.

Example 4.1. Consider an f in $C^*(\mathbb{N})$ such that $f(n) > 0$ for each $n \in \mathbb{N}$ and $lim_{n\to\infty} f(n) = 0$. Then $\beta\mathbb{N} - \mathbb{N} = Z(f^\beta)$. Let $g = f^{-1}$ which belongs to $C(\mathbb{N})$ and set $B = \mathcal{A}(C^*(\mathbb{N}) \cup \{g\})$. Then by Theorem 3.2, $\mathbb{N}$ becomes B-compact. Now define $D = \mathcal{A}(C^*(\mathbb{N}) \cup \{\log_e(1+g)\})$. We shall show that $\mathbb{N}$ is D-compact and $D \subsetneqq B$. Since $lim_{n\to\infty} f(n) = 0$, $lim_{n\to\infty} \frac{\log_e(1+g(n))}{g(n)} = 0$. Consequently $\frac{\log_e(1+g)}{g} \in C^*(\mathbb{N}) \subset B$. Hence $\log_e(1+g)$ is a member of B. Thus D is contained in B. To show that the inclusion relation is proper, we shall show that g belongs to $D - B$. If not, then g must be a polynomial of the members of the set $C^*(\mathbb{N}) \cup \{\log_e(1+g)\}$. This means that g is of the form $g = f_0(\log_e(1+g))^m + f_1(\log_e(1+g))^{m-1} + \cdots + f_m$, where $f_0, f_1, \ldots, f_m \in C^*(\mathbb{N}), m \in \mathbb{N}$. This implies that $\frac{g}{(\log_e(1+g))^n} \in C^*(\mathbb{N})$ — a contradiction to the fact that $lim_{n\to\infty} \frac{g(n)}{\log_e(1+g))^n} = \infty$.

The above example prompts us to frame the following:

Conjecture. There does not exist any minimal subalgebra A of $C(\mathbb{N})$, in the usual inclusion sense, for which $\mathbb{N}$ becomes A-compact.

Nevertheless we give an affirmative answer to Redlin and Watson's question by defining an ordering among the elements of $\Sigma(X)$ in a suitable way, where $\Sigma(X)$ denotes the set of all subalgebras of $C(X)$ containing $C^*(X)$. For each A in $\Sigma(X)$, let α_A be the smallest cardinal number of a subfamily $\mathcal{G}_A$ of $A - C^*(X)$ with the property $A = \mathcal{A}(C^*(X) \cup \mathcal{G}_A)$. For any two A, B of $\Sigma(X)$ we define $A \prec B$ if and only if $\alpha_A \leq \alpha_B$. Then $\prec$ becomes a preorder on $\Sigma(X)$ with respect to which an arbitrary pair of members of $\Sigma(X)$ can be compared.

Theorem 4.2. *Let X be a realcompact space. Then there exists a minimal algebra A_m in $\Sigma(X)$ with respect to the ordering $\prec$ for which X becomes A_m-compact.*

Lemma 4.3. *Given $p \in \beta X$ and $A \in \Sigma(X)$, M_A^p is real if and only if $f^*(p)$ is a real number for each f in A.*

The proof of the lemma is quite similar to that of Theorem 8.4 of [2].

Proof of the theorem. The proof is trivial when X is compact. So suppose that X is not compact. Since X is realcompact, in view of Remark 3.3 we have a subset $\mathcal{F}_m$ of $C^*(X)$ with a smallest cardinal number α with the property $\beta X - X = \bigcup_{f\in\mathcal{F}_m} Z(f^\beta)$ and $Z(f^\beta) \neq \emptyset$ for each $f \in \mathcal{F}_m$. It is clear that each f in $\mathcal{F}_m$ is a unit of $C(X)$ and moreover f^{-1} belongs to $C(X) - C^*(X)$. Let $A_m = \mathcal{A}(C^*(X) \cup \{f^{-1} :$

$f \in \mathcal{F}_m\})$. Then by Theorem 3.2, X is A_m-compact, also $\alpha_{A_m} \leq \alpha$. Assume that for some A in $\Sigma(X)$, X is A-compact. To complete the proof it is enough to show that $\alpha \leq \alpha_A$. Now there exists a subset $\mathcal{G}_A$ of $A - C^*(X)$ with cardinal number α_A such that $A = \mathcal{A}(C^*(X) \cup \mathcal{G}_A)$.

We claim that for each p in $\beta X - X$, there exists a g in $\mathcal{G}_A$ with $g^*(p) = \infty$. If not, then there exists a point q in $\beta X - X$ such that for each h in $\mathcal{G}_A$, $h^*(q)$ is real. Now since X is A-compact and M_A^q is hyperreal, by Lemma 4.3, there exists a g in A for which $g^*(q) = \infty$. Since g can be expressed as $g = t(g_1, g_2, \dots, g_n)$, where $g_1, g_2, \dots, g_n$ are members of $\mathcal{G}_A$ and t is a polynomial in these n variables with coefficients from $C^*(X)$, it follows that $g^*(q)$ is a real number —— a contradiction. Let $\mathcal{F}_A = \{(g \vee \mathbf{1})^{-1} : g \in \mathcal{G}_A\}$; then each member of $\mathcal{F}_A$ is a positive real-valued bounded function on X, taking values arbitrarily near to zero. Therefore in view of the above observation one can write $\beta X - X = \bigcup\{Z(f^\beta) : f \in \mathcal{F}_A\}$ with $Z(f^\beta) \neq \emptyset$ for each f in $\mathcal{F}_A$. Hence by the definition of α, it is less than or equal to the cardinal number of the family $\mathcal{F}_A$ and consequently $\alpha \leq \alpha_A$. □

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