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L_p-NORM UNIFORM DISTRIBUTION

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ABSTRACT. In this paper, the L_p-norm uniform distribution, which is a generalization of the uniform distribution studied by Cambanis, Huang, and Simon (1981), is defined for any $p > 0$. Then its marginal distributions and order statistics are studied.

1. INTRODUCTION

The multivariate normal model is the earliest model derived and is the most commonly used model even today. Most multivariate analysis techniques, e.g. MANOVA, Multivariate Regression, Canonical Analysis, Discriminant Analysis, and Factor Analysis, assume multivariate normal models which in practice may not always be true. So people need richer families of models which may include the normal distribution. As a generalization to the normal model, we have the non-normal models such as spherical models. The L_p-norm spherical distribution has proven useful in Bayesian analysis and robustness studies (e.g. see Kuwana and Kariya [6]), and has also been used by Box and Tiao for the analysis of self- and cross-fertilized data (see Box and Tiao [1]).

In this section, we derive the L_p-norm uniform distribution ($p > 0$), which is used in constructing the L_p-norm spherical distribution. The L_p-norm uniform distribution is developed from the p-generalized normal distribution (Goodman and Kotz [4]) in the same way as the (L_2-norm) uniform distribution was obtained from the normal distribution (Cambanis, Huang, and Simons [2], and Muirhead [7]).

The following lemma will be needed to prove the result.

Lemma 1.1. *Let $\mathbf{x} = (x_1, x_2, \ldots, x_n)'$ $(n \geq 2)$ be an n-vector, $\mathbf{x} \in \mathbb{R}^n \backslash \{0\}$. Consider the transformation*

$$\begin{cases} y_i = x_i/\|\mathbf{x}\|_p, & i = 1, \ldots, n-1, \\ r = \|\mathbf{x}\|_p, \end{cases}$$

where $\|\mathbf{x}\|_p^p = \sum_{i=1}^n |x_i|^p$ and $p > 0$. If $S_1 = \{(x_1, \ldots, x_n) \in \mathbb{R}^n : x_n > 0\}$ and $S_2 = \{(x_1, \ldots, x_n) \in \mathbb{R}^n : x_n < 0\}$, then the Jacobians of the above transformation in S_1 and S_2 are equal and are given by

$$J(\mathbf{x} \to y_1, \ldots, y_{n-1}, r)) = r^{n-1} \left(1 - \sum_{i=1}^{n-1} |y_i|^p\right)^{(1-p)/p}. \tag{1.1}$$

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Proof. Let

$$\Delta(y_i) = \begin{cases} 1, & y_i \geq 0, \\ -1, & y_i < 0. \end{cases}$$

Then $|y_i| = \Delta(y_i)y_i$, $i = 1, \ldots, n-1$. In the region S_i, $i = 1, 2,$, the transformation is 1-1 and we have

$$\begin{cases} x_i = y_i r, & i = 1, \ldots, n-1, \\ x_n = \operatorname{sign}(x_n) r \left(1 - \sum_{i=1}^{n-1} |y_i|^p\right)^{1/p} = \operatorname{sign}(x_n) r \left(1 - \sum_{i=1}^{n-1} [\Delta(y_i)y_i]^p\right)^{1/p}. \end{cases}$$

Then the Jacobian of the transformation in S_i, $i = 1, 2$, is

$$J_i(\mathbf{x} \to y_1, \ldots, y_{n-1}, r) = r^{n-1} \left(1 - \sum_{i=1}^{n-1} |y_i|^p\right)^{1/p-1}.$$

Notice that $J(\mathbf{x} \to y_1, \ldots, y_{n-1}, r)$ does not depend on region S_1 or S_2, so we complete the proof of the lemma. □

Throughout this paper it is assumed that $n \geq 2$ and $p > 0$. Now the derivation of the L_p-norm uniform distribution follows.

Theorem 1.1. *Let* $\mathbf{X}_n = (X_1, \ldots, X_n)'$, *where the* X_i*'s are i.i.d. random variables with p.d.f.*

$$f(x) = \frac{p^{1-1/p}}{2\Gamma(1/p)} e^{-|x|^p/p}, \qquad -\infty < x < \infty.$$

Let $U_i = X_i/\|\mathbf{X}\|_p$, $i = 1, 2, \ldots, n$. *Then* $\sum_{i=1}^n |U_i|^p = 1$ *and the joint p.d.f. of* $U_1, \ldots, U_{n-1}$ *is*

(1.2)

$$p(u_1, \ldots, u_{n-1}) = \frac{p^{n-1}\Gamma(n/p)}{2^{n-1}\Gamma^n(1/p)} \left(1 - \sum_{i=1}^{n-1} |u_i|^p\right)^{(1-p)/p},$$

$$-1 < u_i < 1, \ i = 1, 2, \ldots, n-1, \sum_{i=1}^{n-1} |u_i|^p < 1.$$

Proof. Let $u_i = x_i/\|\mathbf{x}\|_p$, $i = 1, 2, \ldots, n-1$, and $r = \|\mathbf{x}\|_p$, where $-1 < u_i < 1$, $i = 1, \ldots, n-1$, $\sum_{i=1}^{n-1} |u_i|^p < 1$ and $r > 0$. Then the transformation is 1-1 in the regions $S_1 = \{(x_1, \ldots, x_n) \in \mathbb{R}^n : x_n > 0\}$ and $S_2 = \{(x_1, \ldots, x_n) \in \mathbb{R}^n : x_n < 0\}$. According to Lemma 1.1, the Jacobian of the above transformation in S_i, $i = 1, 2$, is given by $r^{n-1}(1 - \sum_1^{n-1} |u_i|^p)^{(1-p)/p}$. Since the p.d.f. of $X_1, \ldots, X_n$ is

$$p(x_1, \ldots, x_n) = \frac{p^{n-n/p}}{2^n\Gamma^n(1/p)} e^{-(1/p)\sum_1^n |x_i|^p}, \qquad -\infty < x_i < \infty, \ i = 1, \ldots, n,$$

the p.d.f. of $U_1, \ldots, U_{n-1}$ and $R = \|\mathbf{X}\|_p$ is

$$p(u_1, \ldots, u_{n-1}, r) = \frac{p^{n-n/p}}{2^{n-1}\Gamma^n(1/p)} r^{n-1} e^{-r^p/p} \left(1 - \sum_{i=1}^{n-1} |u_i|^p\right)^{(1-p)/p}, \quad -1 < u_i < 1,\ i = 1, \ldots, n-1,\ \sum_{1}^{n-1} |u_i|^p < 1,\ r > 0.$$

A straightforward computation will give the joint p.d.f. of $U_1, \ldots, U_{n-1}$ to be

$$p(u_1, \ldots, u_{n-1}) = \frac{p^{n-1}\Gamma(n/p)}{2^{n-1}\Gamma^n(1/p)} \left(1 - \sum_{i=1}^{n-1} |u_i|^p\right)^{(1-p)/p}, \quad -1 < u_i < 1,\ i = 1, \ldots, n-1, \sum_{i=1}^{n-1} |u_i|^p < 1.$$

This completes the proof of the theorem. □

The random vector $\mathbf{U}_n = (U_1, \ldots, U_n)'$ is said to be uniformly distributed on the surface of the (L_p-norm) unit sphere $S_p^n = \{(x_1, \ldots, x_n) \in \mathbb{R}^n : \sum_{i=1}^n |x_i|^p = 1\}$ in $\mathbb{R}^n$. $\mathbf{U}_n$ will be said to have an L_p-norm uniform distribution, denoted by $U(n, p)$. It may be noted that for $p = 2$, $U(n, p)$ becomes the L_2-norm uniform distribution studied by Cambanis, Huang, and Simon [2]. For related work one can refer to Fang, Kotz and Ng [3], and Gupta and Varga [5].

2. Marginal distributions

The marginal densities of $\mathbf{U}_n$ are derived in this section. However, first we give a result on the Jacobian of a transformation.

Lemma 2.1. *Let $x_1, x_2, \ldots, x_n$ be n variables, defined on the whole real line. Let*

$$y_i = |x_i|^p, \qquad i = 1, \ldots, n.$$

Then the Jacobian of this transformation in $S(\sigma_1, \ldots, \sigma_n) = \{(x_1, \ldots, x_n) \in \mathbb{R}^n : \operatorname{sign}(x_i) = \sigma_i,\ i = 1, \ldots, n\}$ is $\frac{1}{p^n} \prod_{i=1}^n y_i^{1/p-1}$, where

$$\sigma_i = \begin{cases} 1, & \text{if } x_i \geq 0, \\ -1, & \text{if } x_i < 0, \end{cases} \qquad i = 1, \ldots, n.$$

Proof. Let $y_i = |x_i|^p$, $i = 1, \ldots, n$. Then the transformation in $S(\sigma_1, \ldots, \sigma_n)$ is 1-1, and $x_i = \operatorname{sign}(x_i) y_i^{1/p}$, $i = 1, \ldots, n$. Hence, the Jacobian in $S(\sigma_1, \ldots, \sigma_n)$, $\sigma_i = \pm 1$, $i = 1, \ldots, n$, is

$$\left|\frac{\partial(\mathbf{x})}{\partial(\mathbf{y})}\right| = \frac{1}{p^n} \prod_{i=1}^n y_i^{1/p-1}, \qquad 0 \leq y_i < \infty,\ i = 1, \ldots, n. \quad \square$$

Theorem 2.1. *If* $\mathbf{U}_n = (U_1, U_2, \dots, U_n)' \sim U(n,p)$, *then*
(1) *the marginal density of* $(U_1, \dots, U_k)'$ $(1 \le k \le n-1)$ *is*

(2.1)
$$p(u_1, \dots, u_k) = \frac{p^k \Gamma(n/p)}{2^k \Gamma^k(1/p)\Gamma((n-k)/p)} \left(1 - \sum_{i=1}^{k} |u_i|^p\right)^{(n-k)/p-1},$$
$$-1 < u_i < 1,\ i = 1, \dots, k, \sum_{i=1}^{k} |u_i|^p < 1;$$

(2) $(|U_1|^p, \dots, |U_k|^p) \sim D_k(\frac{1}{p}, \dots, \frac{1}{p}; \frac{n-k}{p})$, *where* $D_k(\alpha_1, \dots, \alpha_k; \alpha_{k+1})$ *is the Dirichlet distribution with positive parameters* $\alpha_1, \dots, \alpha_k$, α_{k+1}*; and*
(3) $|U_i|^p \sim \mathrm{Beta}(\frac{1}{p}, \frac{n-1}{p})$, $i = 1, \dots, n$, *where* $\mathrm{Beta}(\alpha; \beta)$ *is the Beta distribution with parameters* $\alpha > 0$ *and* $\beta > 0$.

Proof. Part (1) is proven by induction. Let $k = n-1$. Then the p.d.f. of $(U_1, \dots, U_{n-1})$ according to Theorem 1.1 is

$$p(u_1, \dots, u_{n-1}) = \frac{p^{n-1} \Gamma(n/p)}{2^k \Gamma^n(1/p)} \left(1 - \sum_{i=1}^{n-1} |u_i|^p\right)^{(1-p)/p},$$
$$-1 < u_i < 1,\ i - 1, \dots, n-1,\ \sum_{i=1}^{n-1} |u_i|^p < 1.$$

Now, assume (2.1) is true for k. Then the marginal density of $(U_1, \dots, U_{k-1})'$ is given by

$$\begin{aligned} p(u_1, \dots, u_{k-1}) &= \int_{-a}^{a} \frac{p^k \Gamma(n/p)}{2^k \Gamma^k(1/p)\Gamma((n-k)/p)} \left(1 - \sum_{i=1}^{k} |u_i|^p\right)^{(n-k)/p-1} du_k \\ &= \frac{p^k \Gamma(n/p)}{2^{k-1} \Gamma^k(1/p)\Gamma((n-k)/p)} \int_0^a (a^p - u_k^p)^{(n-k)/p-1} du_k \\ &\qquad \left(\text{where } a = \left(1 - \sum_{i=1}^{k-1} |u_i|^p\right)^{1/p}\right) \\ &= \frac{p^{k-1} \Gamma(n/p)}{2^{k-1} \Gamma^{k-1}(1/p)\Gamma((n-(k-1))/p)} \left(1 - \sum_{i=1}^{k-1} |u_i|^p\right)^{(n-(k-1))/p-1}, \\ &\qquad -1 < u_i < 1,\ i = 1, \dots, k-1,\ \sum_{i=1}^{k-1} |u_i|^p < 1, \end{aligned}$$

which means (2.1) is also true for $k-1$. By induction, (2.1) is true for $1 \le k \le n-1$. This completes the proof of part (1).

(2) Let $Z_i = |U_i|^p$, $i = 1, 2, \ldots, k$. Then using the result of part (1) and the Jacobian obtained from Lemma 2.1, we get the p.d.f. of $(Z_1, \ldots, Z_k)$ as

$$\begin{aligned} p(z_1, \ldots, z_k) &= \sum_{(\sigma_1,\ldots,\sigma_k)} \frac{p^k\Gamma(n/p)}{2^k\Gamma^k(1/p)\Gamma((n-k)/p)} \left(1 - \sum_{i=1}^k z_i\right)^{(n-k)/p-1} \left(\frac{1}{p^k}\prod_{i=1}^k z_i^{1/p-1}\right) \\ &= \frac{\Gamma(n/p)}{\Gamma^k(1/p)\Gamma((n-k)/p)} \prod_{i=1}^k z_i^{1/p-1} \left(1 - \sum_{i=1}^k z_i\right)^{(n-k)/p-1}, \\ &\qquad 0 < z_i < 1,\ i = 1, \ldots, k,\ \sum_{i=1}^k z_i < 1, \end{aligned}$$

which is the Dirichlet p.d.f., and hence

$$(|U_1|^p, \ldots, |U_k|^p)' \sim D_k\left(\frac{1}{p}, \ldots, \frac{1}{p}; \frac{n-k}{p}\right).$$

(3) Let $k = 1$ in (2). Then we have $|U_1|^p \sim \mathrm{Beta}(\frac{1}{p}, \frac{n-1}{p})$. But from Theorem 2.1 we know that U_i is defined as $U_i = X_i / \sum_{i=1}^n |X_i|^p$, $i = 1, \ldots, n$, where the X_i's are i.i.d. random variables. Hence $U_i \overset{\mathrm{d}}{=} U_1$, $i = 1, \ldots, n$. Therefore,

$$|U_i|^p \overset{\mathrm{d}}{=} |U_1|^p \sim \mathrm{Beta}\left(\frac{1}{p}, \frac{n-1}{p}\right), \qquad i = 1, \ldots, n. \quad \square$$

3. Order statistics

Some properties of the order statistics of $\mathbf{U}_n$ can be seen in this section.

Theorem 3.1. *Let $\mathbf{U}_n = (U_1, \ldots, U_n)' \sim U(n, p)$. Then the following two results hold.*

(1) $\mathbf{Y} = (Y_1, \ldots, Y_{n-1})' = (U_{(1)}, U_{(2)}, \ldots, U_{(n-1)})'$ *has the p.d.f.*

$$\begin{aligned} p(y_1, \ldots, y_{n-1}) &= \frac{(n-1)!p^{n-1}\Gamma(n/p)}{2^{n-1}\Gamma^n(1/p)} \left(1 - \sum_{i=1}^{n-1} |y_i|^p\right)^{(1-p)/p}, \\ &\qquad -1 < y_1 < y_2 < \cdots < y_{n-1} < 1, \sum_{i=1}^{n-1} |y_i|^p < 1. \end{aligned} \tag{3.1}$$

(2) *The $(n-1)$-dimensional random vector $(W_1, \ldots, W_{n-1})'$, where*

$$\begin{cases} W_1^p = (n-1)|Y_1|^p, \\ W_2^p = (n-2)|Y_2|^p - |Y_1|^p), \\ \vdots \\ W_{n-1}^p = |Y_{n-1}|^p - |Y_{n-2}|^p, \end{cases} \tag{3.2}$$

has the p.d.f.

$$
\begin{aligned}
p(w_1,\ldots,w_{n-1}) &= \frac{(n-1)!p^{n-1}\Gamma(n/p)}{\sqrt[p]{n-1}\Gamma^n(1/p)}\left[\prod_{i=2}^{n-1}\frac{w_i^{p-1}}{n-i}\sum_{j=1}^{i}\left(\frac{w_j^p}{n-j}\right)^{1/p-1}\right] \\
&\cdot\left(1-\sum_{i=1}^{n-1}w_i^p\right)^{(1-p)/p}, \quad 0<w_i<1,\ i=1,\ldots,n-1,\ \sum_{i=1}^{n-1}w_i^p<1.
\end{aligned} \tag{3.3}
$$

Proof. (1) Define $y_i = u_{(i)}$, $i=1,\ldots,n-1$ and $S_{\pi_i} = \{(u_1,\ldots,u_{n-1})\in\mathbb{R}^{n-1} : u_{i_1}<\cdots<u_{i_{n-1}}\}$, where $(i_1,\ldots,i_{n-1})$ is a permutation of $(1,2,\ldots,n-1)$. In S_{π_i}, $y_j = u_{ij}$, $j=1,\ldots,n-1$. So the Jacobian $J_{\pi_i}(u_1,\ldots,u_{n-1}\to y_1,\ldots,y_{n-1})$ of the transformation is 1. Hence, the joint of p.d.f. of $(Y_1,\ldots,Y_{n-1})$ is

$$
\begin{aligned}
&p(y_1,\ldots,y_{n-1}) \\
&= \sum_{S_{\pi_i}}\frac{p^{n-1}\Gamma(n/p)}{2^{n-1}\Gamma^n(1/p)}\left(1-\sum_{i=1}^{n-1}|y_i|^p\right)^{(1-p)/p} J_{\pi_i}(u_1,\ldots,u_{n-1}\to y_1,\ldots,y_{n-1}) \\
&= \frac{(n-1)!p^{n-1}\Gamma(n/p)}{2^{n-1}\Gamma^n(1/p)}\left(1-\sum_{i=1}^{n-1}|y_i|^p\right)^{(1-p)/p}, \\
&\qquad\qquad -1<y_1<\cdots<y_{n-1}<1,\ \sum_{i=1}^{n-1}|y_i|^p<1.
\end{aligned}
$$

(2) Define

$$
\begin{cases}
w_1^p = (n-1)|y_1|^p, \\
w_2^p = (n-2)(|y_2|^p-|y_1|^p), \\
\vdots \\
w_{n-1}^p = |y_{n-1}|^p - |y_{n-2}|^p.
\end{cases}
$$

Then $\sum_{i=1}^{n-1}w_i^p = \sum_{i=1}^{n-1}|y_i|^p$ and the transformation is 1-1 in the region $S(\sigma_1,\ldots,\sigma_{n-1}) = \{(y_1,\ldots,y_{n-1})\in\mathbb{R}^{n-1} : \mathrm{sign}(y_i)=\sigma_i,\ i=1,\ldots,n-1\}$, where

$$
\sigma_i = \begin{cases} 1, & \text{if } y_i\ge 0, \\ -1, & \text{if } y_i<0, \end{cases} \qquad i=1,\ldots,n-1.
$$

It can be shown that the Jacobian of the transformation (3.2) in $S(\sigma_1,\ldots,\sigma_{n-1})$ is

$$
J(y_1,\ldots,y_{n-1}\to w_1,\ldots,w_{n-1}) = \frac{1}{\sqrt[p]{n-1}}\prod_{i=2}^{n-1}\left[\frac{w_i^{p-1}}{n-i}\sum_{j=1}^{i}\left(\frac{w_j^p}{n-j}\right)^{1/p-1}\right]
$$

for every $(\sigma_1, \dots, \sigma_{n-1})$, $\sigma_i = \pm 1$, $i = 1, \dots, n-1$. Therefore, the joint p.d.f. of $(W_1, \dots, W_{n-1})$ is

$$\begin{aligned}
&p(w_1, \dots, w_{n-1}) \\
&\quad = \sum_{(\sigma_1,\dots,\sigma_{n-1})} \Bigg\{ \left[\frac{(n-1)!p^{n-1}\Gamma(n/p)}{2^{n-1}\Gamma^n(1/p)} \left(1 - \sum_{i=1}^{n-1} w_i^p\right)^{(1-p)/p} \right] \\
&\qquad\qquad \cdot \frac{1}{\sqrt[p]{n-1}} \prod_{i=2}^{n-1} \left[\frac{w_i^{p-1}}{n-i} \sum_{j=1}^{i} \left(\frac{w_j^p}{n-j}\right)^{1/p-1} \right] \Bigg\} \\
&\quad = \frac{(n-1)!p^{n-1}\Gamma(n/p)}{\sqrt[p]{n-1}\,\Gamma^n(1/p)} \left[\prod_{i=2}^{n-1} \frac{w_i^{p-1}}{n-i} \sum_{j=1}^{i} \left(\frac{w_j^p}{n-j}\right)^{1/p-1} \right] \\
&\qquad \cdot \left(1 - \sum_{i=1}^{n-1} w_i^p\right)^{(1-p)/p}, \\
&\qquad\qquad w_i > 0,\ i = 1, 2, \dots, n-1,\ \sum_{i=1}^{n-1} w_i^p < 1. \quad \square
\end{aligned}$$

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