FIXED POINT THEORY FOR COMPACT UPPER SEMI-CONTINUOUS OR LOWER SEMI-CONTINUOUS SET VALUED MAPS

DONAL O'REGAN

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ABSTRACT. Fixed point theory is presented for compact u.s.c. and l.s.c. set valued maps.

1. INTRODUCTION

This paper presents new fixed point theory for compact upper semi-continuous (u.s.c.) and lower semi-continuous (l.s.c.) set valued mappings. Our results generalize some well known fixed point results for convex and nonconvex maps. Our paper is divided into two main sections. In section 2 we use a recent Leray–Schauder type alternative due to Ben–El–Mechaiekh and Idzik [3] for u.s.c., compact approachable maps with nonempty, compact values to establish a new fixed point result for such maps. Our theory generalizes some fixed point results in Ben–El–Mechaiekh and Deguire [2] and Granas [9]. In section 3 we establish via Michael's selection theorem [11] a nonlinear alternative of Leray–Schauder type for l.s.c., compact multifunctions with closed, convex values. Also a new fixed point result is given. This generalizes some results in Himmelberg, Porter and Van Vleck [10] and Tarafdar and V'ýborn'ý [17]. A brief discussion of nonconvex maps is also given.

We now gather together some definitions and known facts. Let X and Y be topological spaces. A multifunction $f : X \rightarrow Y$ is a point to set function such that for each $x \in X$, $f(x)$ is a nonempty subset of Y . The function f is u.s.c. if the set $f^{-1}(B) = \{x \in X : f(x) \cap B \neq \emptyset\}$ is closed for any closed set B in Y (equivalently, $f : X \rightarrow Y$ is u.s.c. if for any net $\{x_\alpha\}$ in X and any closed set B in Y with $x_\alpha \rightarrow x_0 \in X$ and $f(x_\alpha) \cap B \neq \emptyset$ for all α we have $f(x_0) \cap B \neq \emptyset$). The function $f : X \rightarrow Y$ is l.s.c. if the set $f^{-1}(A)$ is open for any open set A in Y . $f : X \rightarrow Y$ is compact if $f(X)$ is relatively compact in Y . Finally the function f is said to be point-compact if for each $x \in X$, $f(x)$ is a compact subset of Y . Now we state a result [16, proposition 1] which will be used frequently in this paper.

Theorem 1.1. *Let X and Y be topological spaces and $f : X \rightarrow Y$ be a u.s.c., point-compact multifunction. Suppose $\{x_\alpha\}$ is a net in X such that $x_\alpha \rightarrow x_0$. If $y_\alpha \in f(x_\alpha)$ for each α , then there is a $y_0 \in f(x_0)$ and a subnet $\{y_\beta\}$ of the net $\{y_\alpha\}$ such that $y_\beta \rightarrow y_0$.*

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Theorem 1.1 immediately yields the following well known result.

Theorem 1.2. *Let X and Y be topological spaces and $f : X \rightarrow Y$ be a u.s.c., point-compact multifunction. Suppose $\{x_\alpha\}$ is a net in X and $y_\alpha \in f(x_\alpha)$ for each α . If $x_\alpha \rightarrow x_0$ and $y_\alpha \rightarrow y_0$, then $y_0 \in f(x_0)$.*

2. THE UPPER SEMI-CONTINUOUS CASE

We begin by describing the class of approachable set valued maps. Let X and Y be subsets of Hausdorff topological vector spaces E_1 and E_2 respectively, and let $F : X \rightarrow Y$ be a multifunction. Given two open neighborhoods U and V of the origins in E_1 and E_2 respectively, a (U, V) -continuous selection of F is a continuous function $s : X \rightarrow Y$ satisfying

$$s(x) \in (F[(x+U) \cap X] + V) \cap Y \quad \text{for every } x \in X.$$

F is said to be approachable if it has a (U, V) -continuous selection for any open neighborhoods U and V of the origins in E_1 and E_2 respectively. Let

$$\mathcal{A}_0(X, Y) = \{F : X \rightarrow Y : F \text{ is approachable}\}$$

and

$$\mathcal{A}(X, Y) = \{F \in \mathcal{A}_0(X, Y) : F \text{ is u.s.c. and compact valued}\}.$$

Remark. For examples of approachable maps we refer the reader to [2, 3].

We now gather together three theorems from the literature which will be used in this section.

Theorem 2.1 ([2], pg.486, proposition 3.3). *Let X, Y and Z be subsets of the Hausdorff topological vector spaces E_1, E_2 and E_3 respectively. Suppose $F : X \rightarrow Y$ is a multivalued map with $F \in \mathcal{A}(X, Y)$ and $r : Z \rightarrow X$ is a continuous map (single valued). If Z is compact then $Fr \in \mathcal{A}(Z, Y)$.*

Theorem 2.2 ([2], pg.496, corollary 7.3). *Let X be a convex subset of a locally convex Hausdorff linear topological space E with $F \in \mathcal{A}(X, X)$ a compact map. Then F has a fixed point (i.e. there exists $x \in X$ with $x \in F(x)$).*

Theorem 2.3 ([3], pg.108, Theorem). *Let X be a closed subset of a locally convex Hausdorff linear topological space E with $0 \in \text{int } X$. In addition assume $F : X \rightarrow E$ is a u.s.c., approachable, compact map with nonempty, closed values. Then either*

- (A1) F has a fixed point in X ; or
- (A2) there exists $(\lambda, x) \in (0, 1) \times \partial X$ such that $x \in \lambda F(x)$.

We now state and prove the main result of this section.

Theorem 2.4. *Let C be a complete convex subset of a metrizable (metric d) locally convex linear topological space E with Q a closed, convex, proper subset of C and $0 \in Q$. Assume $F : Q \rightarrow C$ is a u.s.c., approachable, compact map with nonempty, compact values. In addition suppose*

$$(2.1) \quad \begin{cases} \text{if } \{(x_j, \lambda_j)\}_{j=1}^\infty \text{ is a sequence in } \partial Q \times [0, 1] \text{ converging to} \\ (x, \lambda) \text{ with } x \in \lambda F(x) \text{ and } 0 \leq \lambda < 1, \\ \text{then there exists } j_0 \in \{1, 2, \dots\} \text{ with} \\ \{\lambda_j F(x_j)\} \subseteq Q \text{ for each } j \geq j_0. \end{cases}$$

Finally assume either

$$(2.2a) \quad \begin{cases} C \text{ is compact and } U_i = \{x \in E : d(x, Q) < \frac{1}{i}\} \subseteq C \\ \text{for } i \text{ sufficiently large,} \end{cases}$$

or

$$(2.2b) \quad Q \subseteq \overline{\text{co}}(\overline{F(Q)}) \text{ and } U_i \subseteq \overline{\text{co}}(\overline{F(Q)}) \text{ for } i \text{ sufficiently large,}$$

or

$$(2.2c) \quad \begin{cases} Fr \in \mathcal{A}_0(C, C) \text{ and } U_i \subseteq C \text{ for } i \text{ sufficiently large; here} \\ r : E \rightarrow Q \text{ is a continuous retraction such that} \\ r(z) \in \partial Q \text{ for } z \in E/Q \end{cases}$$

is satisfied. Then F has a fixed point in Q .

Remarks. (i) The existence of a continuous retraction $r : E \rightarrow Q$ follows immediately from Dugundji's extension theorem [7, 18].

(ii) If $0 \in \text{int } Q$ we may take

$$r(x) = \frac{x}{\max\{1, \mu(x)\}}, \quad x \in E.$$

Here μ is the Minkowski functional [15] on Q , i.e. $\mu(x) = \inf\{\alpha > 0 : x \in \alpha Q\}$. If $\text{int } Q = \emptyset$ then $\partial Q = Q$. As a result we may (and do) choose the retraction r above so that $r(z) \in \partial Q$ if $z \in E/Q$.

(iii) Notice (2.2a) and (2.2b) are only of interest if the space is normable.

(iv) If F is convex valued then of course Fr is convex valued, so Fr is approachable [3], i.e. (2.2c) holds.

(v) Theorem 2.4 was proved by Furi and Pera [8], by a different method, when F is single valued, compact and continuous. It was extended by O'Regan [13, 14] for the case when F is single valued, condensing (P -concentrative) and continuous.

Proof. Let $Z = C$ if (2.2a) or (2.2c) holds, and let $Z = \overline{\text{co}}(\overline{F(Q)})$ if (2.2b) is satisfied.

Remark. Note $\overline{\text{co}}(\overline{F(Q)})$ is compact. To see this let P be a defining system of seminorms. Fix $p \in P$. Since $\overline{F(Q)}$ is compact then $\overline{\text{co}}(\overline{F(Q)})$ is precompact (totally bounded) [6] in the seminormed space (E, p) for each $p \in P$. Thus $\overline{\text{co}}(\overline{F(Q)})$ is precompact in E . Now since $\overline{\text{co}}(\overline{F(Q)})$ is a closed precompact subset of C and C is complete, then $\overline{\text{co}}(\overline{F(Q)})$ is compact [5].

Now let

$$B = \{x \in Z : x \in Fr(x)\}.$$

We first show $B \neq \emptyset$. Notice $Fr : Z \rightarrow Z$ is a u.s.c. (since F is u.s.c. and r is continuous), approachable (theorem 2.1 if (2.2a) or (2.2b) occurs and by assumption if (2.2c) occurs), compact map with nonempty, compact values. Theorem 2.2 implies that Fr has a fixed point, so $B \neq \emptyset$. We next claim that B is closed. Let (x_α) be a net in B with $x_\alpha \rightarrow x_0 \in Z$. Theorem 1.2 implies $x_0 \in Fr(x_0)$, so $x_0 \in B$. Thus B is closed, and in fact B is compact since $B \subseteq Fr(B) \subseteq F(Q)$.

It remains to show $B \cap Q \neq \emptyset$. To do this we argue by contradiction. Suppose $B \cap Q = \emptyset$. Then since B is compact and Q is closed there exists $\delta > 0$ with $\text{dist}(B, Q) > \delta$. Choose $m \in \{1, 2, \dots\}$ such that $1 < \delta m$ and $U_i \subseteq Z$ for $i \in \{m, m+1, \dots\}$; here $U_i = \{x \in E : d(x, Q) < \frac{1}{i}\}$. Fix $i \in \{m, m+1, \dots\}$. Since $\text{dist}(B, Q) > \delta$, then $B \cap \overline{U_i} = \emptyset$. Now $0 \in U_i$ and $Fr : \overline{U_i} \rightarrow C$ is a u.s.c., approachable, compact map with nonempty, compact values. Theorem 2.3 implies (since $B \cap \overline{U_i} = \emptyset$) that there exists $(y_i, \lambda_i) \in \partial U_i \times (0, 1)$ with $y_i \in \lambda_i Fr(y_i)$. Notice in particular since $y_i \in \partial U_i$ that

$$(2.3) \quad \{\lambda_i Fr(y_i)\} \not\subseteq Q \quad \text{for each } i \in \{m, m+1, \dots\}.$$

Now consider

$$D = \{x \in E : x \in \lambda Fr(x) \text{ for some } \lambda \in [0, 1]\}.$$

We first show D is closed. To see this let (x_α) be a net in D (i.e. for each α , $x_\alpha \in \lambda_\alpha Fr(x_\alpha)$ for some $\lambda_\alpha \in [0, 1]$) with $x_\alpha \rightarrow x_0 \in E$. Without loss of generality assume $\lambda_\alpha \rightarrow \lambda_0 \in [0, 1]$. We claim that

$$N : E \times [0, 1] \rightarrow C \quad \text{given by} \quad N(x, \lambda) = \lambda Fr(x)$$

is u.s.c. with nonempty, compact values. If the claim is true then theorem 1.2 implies $x_0 \in N(x_0, \lambda_0)$, i.e. $x_0 \in \lambda_0 Fr(x_0)$, so D is closed. To see that N is u.s.c. let Ω be a closed subset of C , (y_α, t_α) a net in $E \times [0, 1]$, $(y_\alpha, t_\alpha) \rightarrow (y_0, t_0)$ and $t_\alpha Fr(y_\alpha) \cap \Omega \neq \emptyset$. We must show $t_0 Fr(y_0) \cap \Omega \neq \emptyset$. Suppose $w_\alpha \in Fr(y_\alpha)$ with $t_\alpha w_\alpha \in \Omega$. Since Fr is u.s.c., there exist (theorem 1.1) $w_0 \in Fr(y_0)$ and a subnet (w_β) of (w_α) with $w_\beta \rightarrow w_0$. Since Ω is closed we have $t_0 w_0 \in \Omega$. Consequently $t_0 Fr(y_0) \cap \Omega \neq \emptyset$, so $N : E \times [0, 1] \rightarrow C$ is u.s.c. Thus D is closed and also compact since $D \subseteq co(F(Q) \cup \{0\})$. This together with $d(y_j, Q) = \frac{1}{j}$, $|\lambda_j| \leq 1$ (for $j \in \{m, m+1, \dots\}$), implies that we may assume without loss of generality that $\lambda_j \rightarrow \lambda^*$ and $y_j \rightarrow y^* \in \partial Q$. Also, since $y_j \in \lambda_j Fr(y_j)$, we have from theorem 1.2 that $y^* \in \lambda^* Fr(y^*)$ (note from above that $N : \overline{U_m} \times [0, 1] \rightarrow C$ given by $N(u, \lambda) = \lambda Fr(u)$ is u.s.c. with nonempty, compact values). If $\lambda^* = 1$, then $y^* \in Fr(y^*)$, which contradicts $B \cap Q = \emptyset$. Hence we may assume $0 \leq \lambda^* < 1$. But in this case, (2.1) with $x_j = r(y_j) \in \partial Q$, $x = y^* = r(y^*)$ implies that there exists $j_0 \in \{1, 2, \dots\}$ with $\{\lambda_j Fr(y_j)\} \subseteq Q$ for each $j \geq j_0$. This contradicts (2.3). Thus $B \cap Q \neq \emptyset$, so there exists $x \in Q$ with $x \in Fr(x)$, i.e. $x \in F(x)$. $\square$

Let X and Y be subsets of a locally convex Hausdorff linear topological space E . More generally, let

$$\mathcal{B}_0(X, Y) = \{F : X \rightarrow Y : F \text{ satisfies property } B\}$$

and let $\mathcal{B}(X, Y)$ be the set of maps $F : X \rightarrow Y$ which are u.s.c., compact with nonempty, compact values and satisfy a property B (which we will not specify). Suppose the following properties are satisfied:

$$(2.4) \quad \text{if } X \text{ is convex and } F \in \mathcal{B}(X, Y) \text{ then } F \text{ has a fixed point in } X$$

and

$$(2.5) \quad \begin{cases} \text{if } X \text{ is closed and convex, } 0 \in \text{int } X \text{ and } F \in \mathcal{B}(X, E) \\ \text{then either} \\ (A1). F \text{ has a fixed point in } X; \text{ or} \\ (A2). \text{ there exists } (\lambda, x) \in (0, 1) \times \partial X \text{ such that } x \in \lambda F(x). \end{cases}$$

With the above properties, essentially the same reasoning as in theorem 2.4 establishes the following result.

Theorem 2.5. *Let C be a complete convex subset of a metrizable (metric d) locally convex linear topological space E with Q a closed, convex, proper subset of C , $0 \in Q$ and $U_i = \{x \in E : d(x, Q) < \frac{1}{i}\} \subseteq C$ for i sufficiently large. Assume $F \in \mathcal{B}(Q, C)$, (2.1) holds and also that*

$$(2.6) \quad \begin{cases} Fr \in \mathcal{B}_0(C, C); \text{ here } r : E \rightarrow Q \text{ is a continuous retraction} \\ \text{such that } r(z) \in \partial Q \text{ for } z \in E/Q \end{cases}$$

is satisfied. Then F has a fixed point in Q .

Remark. For examples of classes of maps $\mathcal{B}_0(X, Y)$ we refer the reader to [9]. For example, property B could be that the values of the maps are R_δ sets.

3. THE LOWER SEMI-CONTINUOUS CASE

We begin by establishing a nonlinear alternative of Leray–Schauder type for l.s.c. set valued maps.

Theorem 3.1. *Let Q be either (i) a closed convex subset of a Fréchet space (complete metrizable locally convex linear topological space) E , or (ii) a complete, convex, compact, metrizable subset of a locally convex Hausdorff linear topological space E . Assume U is a relatively open subset of Q , $0 \in U$ and U is convex. Suppose $F : \overline{U} \rightarrow Q$ is a l.s.c., compact map with closed, convex values. Then either*

- (A1) *F has a fixed point in $\overline{U}$; or*
- (A2) *there exist $u \in \partial U$ and $\lambda \in (0, 1)$ with $u \in \lambda F(u)$.*

Remark. $\overline{U}$ and ∂U denote the closure and boundary of U in Q respectively.

Proof. Define the retraction $r : Q \rightarrow \overline{U}$ by

$$r(x) = \frac{x}{\max\{1, \mu(x)\}}, \quad x \in Q,$$

where μ is the Minkowski functional [15] on $\overline{U}$, i.e. $\mu(x) = \inf\{\alpha > 0 : x \in \alpha \overline{U}\}$. Consider the mapping $Fr : Q \rightarrow Q$. Notice Fr is an l.s.c., compact map with closed, convex values. Michael's selection theorem [11, theorem 1.2] implies that there exists a continuous function $f : Q \rightarrow Q$ such that $f(x) \in Fr(x)$ for all $x \in Q$. Next Tychonoff's theorem [7] implies that f has a fixed point, which is also of course a fixed point of Fr , i.e. there exists $y \in Q$ with $y \in Fr(y)$. Thus if $z = r(y) \in \overline{U}$ we have $z \in rF(z)$, i.e. $z = r(w)$ for some $w \in F(z)$, i.e. $z \in \overline{U}$ is a fixed point of rF . Now either $w \in \overline{U}$ or not. If $w \in \overline{U}$ then $r(w) = w$, so (A1) occurs. If $w \notin \overline{U}$ then $z = r(w) = \frac{w}{\mu(w)}$, so $z \in \lambda F(z)$ where $\lambda = \frac{1}{\mu(w)} < 1$, i.e. (A2) occurs. $\square$

Remark. In theorem 3.1 conditions were put on Q, E and F so that Michael's selection theorem could be applied. Of course results could be established if we used other selection theorems for l.s.c. set valued maps (convex and nonconvex); see [1, 4, 12] for example. This remark also applies to the next theorem.

We now prove a new fixed point result for l.s.c., compact set valued maps.

Theorem 3.2. *Let C be either (i) a closed convex subset of a Fréchet space (metric d) E or (ii) a complete, convex, compact subset of a metrizable (metric d) locally convex linear topological space E . Assume Q is a closed, convex, proper subset of C with $0 \in Q$ and $U_i = \{x \in E : d(x, Q) < \frac{1}{i}\} \subseteq C$ for i sufficiently large. Suppose $F : Q \rightarrow C$ is an l.s.c., compact map with closed, convex values, and that (2.1) holds. Then F has a fixed point.*

Proof. Let $r : E \rightarrow Q$ be a continuous retraction with $r(z) \in \partial Q$ for $z \in E/Q$. By Michael's selection theorem [11] (since $Fr : C \rightarrow C$ is an l.s.c., compact map with closed, convex values) there exists a continuous function $f : C \rightarrow C$ such that $f(x) \in Fr(x)$ for all $x \in C$. Consider

$$B = \{x \in C : x = f(x)\}.$$

Now $B \neq \emptyset$ by Tychonoff's theorem. Also B is clearly closed and compact, since $B \subseteq f(B) \subseteq F(Q)$. It remains to show $B \cap Q \neq \emptyset$. Suppose $B \cap Q = \emptyset$. Then there exists $\delta > 0$ with $\text{dist}(B, Q) > \delta$. Choose $m \in \{1, 2, \dots\}$ such that $1 < \delta m$ and $U_i \subseteq C$ for $i \in \{m, m+1, \dots\}$. Fix $i \in \{m, m+1, \dots\}$. Since $\text{dist}(B, Q) > \delta$, then $B \cap \overline{U_i} = \emptyset$. Now $f : \overline{U_i} \rightarrow C$ is a continuous, compact map, so by the nonlinear alternative for single valued maps [7] there exists (since $B \cap \overline{U_i} = \emptyset$) $(y_i, \lambda_i) \in \partial U_i \times (0, 1)$ with $y_i = \lambda_i f(y_i)$. Now since $y_i \in \partial U_i$ we have $\lambda_i f(y_i) \notin Q$ for $i \in \{m, m+1, \dots\}$, and so

$$(3.1) \quad \{\lambda_i Fr(y_i)\} \not\subseteq Q \quad \text{for each } i \in \{m, m+1, \dots\}.$$

Let

$$D = \{x \in C : x = \lambda f(x) \text{ for some } \lambda \in [0, 1]\}.$$

Since D is compact and $d(y_j, Q) = \frac{1}{j}$, $|\lambda_j| \leq 1$ (for $j \in \{m, m+1, \dots\}$), we may assume without loss of generality that $\lambda_j \rightarrow \lambda^*$ and $y_j \rightarrow y^* \in \partial Q$. Also, since $y_j = \lambda_j f(y_j)$ we have $y^* = \lambda^* f(y^*)$. Now $\lambda^* \neq 1$ since $B \cap Q = \emptyset$. Thus $0 \leq \lambda^* < 1$ and $y^* \in \lambda^* Fr(y^*)$. But in this case (2.1) with $x_j = r(y_j)$, $x = y^* = r(y^*)$ implies that there exists $j_0 \in \{1, 2, \dots\}$ with $\{\lambda_j Fr(y_j)\} \subseteq Q$ for each $j \geq j_0$. This contradicts (3.1), so $B \cap Q \neq \emptyset$. $\square$

Remark. One could use the ideas in this section together with ideas in [14] to obtain new results of Leray–Schauder and Furi–Pera type for l.s.c., condensing (P -concentrative) maps.

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DEPARTMENT OF MATHEMATICS, UNIVERSITY COLLEGE GALWAY, GALWAY, IRELAND

E-mail address: Donal.ORegan@UCG.IE