QUASITILTED ALGEBRAS ADMIT A PREPROJECTIVE COMPONENT

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ABSTRACT. Quasitilted algebras are generalizations of tilted algebras. As a main result we show here that the Auslander-Reiten quiver of such an algebra has a preprojective component

Let k be an algebraically closed field and Λ be a finite-dimensional k -algebra. We denote by $\text{mod } \Lambda$ the category of finitely generated left Λ -modules. By Γ_Λ we denote the Auslander-Reiten quiver of Λ . Recall that the vertices of Γ_Λ correspond to the isomorphism classes of indecomposable finitely generated Λ -modules. The number of arrows from an indecomposable Λ -module X to an indecomposable Λ -module Y is the dimension of the k -vector space $\text{rad}(X, Y)/\text{rad}^2(X, Y)$, where $\text{rad}(-, -)$ denotes the Jacobson radical of $\text{mod } \Lambda$. We denote by $\tau X = DTr X$ the Auslander-Reiten translate of the indecomposable Λ -module X . This is defined for each indecomposable module, and in case X is non-projective the translate τX will be indecomposable and non-injective. Dually there is defined $\tau^- X = TrDX$. A connected component $\mathcal{P}$ of Γ_Λ is called a *preprojective component* if $\mathcal{P}$ does not contain an oriented cycle and each $X \in \mathcal{P}$ is of the form $\tau^{-r} P$ for some $r \in \mathbb{N}$ and an indecomposable projective Λ -module P . For details see [ARS]. The existence of preprojective components has been established for various classes of algebras such as tilted algebras [St] or algebras satisfying the separation condition [B]. It is well known that tilted algebras may have several preprojective components. One of the important features of an indecomposable module X lying in a preprojective component is that X is homologically trivial, i.e. $\text{Ext}_\Lambda^i(X, X) = 0$ for $i > 0$ and $\text{End}_\Lambda X = k$ and that its isomorphism class is uniquely determined by the composition factors. Also these modules can be constructed easily by the so-called knitting procedure.

Quasitilted algebras have been introduced and investigated in [HRS2]. Recall that a finite-dimensional k -algebra Λ is called a *quasitilted algebra* if there exist a hereditary abelian k -category $\mathcal{H}$ and a tilting object $T \in \mathcal{H}$ such that $\Lambda = \text{End}_{\mathcal{H}} T$. In this article we will not work with this definition but rather with the homological characterization established in [HRS2]. We will use the following notation. For

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$X \in \text{mod } \Lambda$ we denote by $\text{pd}_\Lambda X$ (resp. $\text{id}_\Lambda X$) the *projective dimension* (resp. the *injective dimension*) of X , and we denote by $\text{gl.dim } \Lambda$ the *global dimension* of Λ . The algebra Λ is quasitilted if and only if $\text{gl.dim } \Lambda = 2$, and for each indecomposable Λ -module X we have either $\text{pd}_\Lambda X \leq 1$ or $\text{id}_\Lambda X \leq 1$.

As a main result we will show that the Auslander-Reiten quiver of a quasitilted algebra always has a preprojective component. Since tilted algebras are known to have a preprojective component we can restrict to the case of quasitilted algebras which are not tilted. In this situation we can use results from [CS], where also other aspects of the structure of the Auslander-Reiten quiver of a quasitilted algebra are considered. We also refer to [S] for recent results on tame quasitilted algebras.

The key idea of the proof is to investigate in detail conditions on a module M over a quasitilted algebra Λ such that the one-point extension algebra $\Lambda[M]$ is again quasitilted. This will extend results obtained in [HRS2]. The results obtained here will then allow us to make use of a result in [DP].

In the first section we start by recalling some preliminary facts. The second section contains the proof of the theorem, while in section 3 we will present some examples. We denote the composition of morphisms $f : X \rightarrow Y$ and $g : Y \rightarrow Z$ in a given category $\mathcal{K}$ by fg . The following notation will be useful. Let M, N, X be Λ -modules and $f : M \rightarrow X$ a map. Then we will denote by $\hat{f} : M \oplus N \rightarrow X$ the map whose restriction to M is f and whose restriction to N is zero. For unexplained terminology and for some representation-theoretic background we refer to [ARS] and [R].

1. PRELIMINARIES

In this section we will recall some basic facts on quasitilted algebras from [HRS2]. Moreover, we will study one-point extensions of quasitilted algebras.

Let Λ be a finite-dimensional k -algebra. A *path* in $\text{mod } \Lambda$ is a sequence $(X_0, \dots, X_t)$ of indecomposable Λ -modules X_i for $0 \leq i \leq t$, such that there is a map $0 \neq f_i \in \text{rad}(X_i, X_{i+1})$ for $0 \leq i < t$. In this case we write $X_0 \preccurlyeq X_t$ and say that X_0 is a predecessor of X_t and that X_t is a successor of X_0 . If $t \geq 1$ and $X_0 \simeq X_t$ we say that the path is a *cycle*. If $t = 1$ and $X_0 \simeq X_1$ we say that the path is a *short cycle*. We say that the path $(X_0, \dots, X_t)$ is *sectional* if $X_{i-1} \not\preccurlyeq \tau X_{i+1}$ for $0 < i < t$. If $(X_0, \dots, X_t)$ is a path, we say that a path $(Y_0, \dots, Y_s)$ is a *refinement* of $(X_0, \dots, X_t)$ if there is an order-preserving function $\pi : \{0, \dots, t\} \rightarrow \{0, \dots, s\}$ such that $X_i = Y_{\pi(i)}$, $\pi(0) = 0$, $\pi(t) = s$. A refinement $(Y_0, \dots, Y_s)$ of a path $(X_0, \dots, X_t)$ is said to be a *refinement of irreducible maps* if there is an irreducible map from Y_i to Y_{i+1} for all $0 \leq i < s$, or equivalently $\text{rad}(Y_i, Y_{i+1})/\text{rad}^2(Y_i, Y_{i+1}) \neq 0$ for all $0 \leq i < s$.

Following [HR] we say that a module M is *directing* provided there do not exist indecomposable summands M_1 and M_2 of M , and an indecomposable non-projective module W such that $M_1 \preccurlyeq \tau W$ and $W \preccurlyeq M_2$. We refer to [HR] for a further study of directing modules.

The following subcategories of $\text{mod } \Lambda$ are useful. We denote by $\text{ind } \Lambda$ the full subcategory of $\text{mod } \Lambda$ containing a chosen set of representatives of the isomorphism classes of indecomposable Λ -modules. We denote by $\mathcal{L} = \mathcal{L}(\Lambda)$ the full subcategory of $\text{ind } \Lambda$ containing those indecomposable modules X such that every predecessor Y of X satisfies $\text{pd}_\Lambda Y \leq 1$. Dually, we denote by $\mathcal{R} = \mathcal{R}(\Lambda)$ the full subcategory of $\text{ind } \Lambda$ containing those indecomposable modules X such that every successor Y of X

satisfies $\text{id}_\Lambda Y \leq 1$. Using this we have the following characterization of quasitilted algebras [HRS2], Theorem 1.14.

Theorem 1.1. *The following are equivalent for a finite-dimensional k -algebra Λ .*

- (i) Λ is quasitilted.
- (ii) $\mathcal{L}$ contains all indecomposable projective modules.
- (iii) $\mathcal{R}$ contains all indecomposable injective modules.
- (iv) Any path in $\text{mod } \Lambda$ starting in an indecomposable injective module and ending in an indecomposable projective module has a refinement of irreducible maps, and any such refinement is sectional.

We will also need the following lemma. A proof is basically contained in [HRS1] or [HRS2].

Lemma 1.2. *Let Λ be a quasitilted algebra and $(X_0, \dots, X_t)$ be a path contained in $\text{ind } \Lambda$. If X_0 belongs to $\mathcal{R}$ or if X_t belongs to $\mathcal{L}$, then there exist an indecomposable module Y and nonzero maps $X_0 \rightarrow Y$ and $Y \rightarrow X_t$. In particular, an indecomposable Λ -module X belongs to a cycle if and only if it belongs to a short cycle.*

We need the notion of a one-point extension algebra. Let Λ be a finite-dimensional k -algebra and M in $\text{mod } \Lambda$. The one-point extension algebra $\Lambda[M]$ of Λ by M is by definition the finite dimensional k -algebra

$$\Lambda[M] = \begin{pmatrix} k & 0 \\ M & \Lambda \end{pmatrix}.$$

If $\Delta = \Lambda[M]$ is the one-point extension algebra of Λ by M then the category of Δ -modules is equivalent to the category of triples $(k^t, {}_\Lambda X, f)$, where $f : M \otimes k^t \rightarrow X$ is a map of Λ -modules.

It was shown in [HRS2] that a quasitilted algebra Δ is always of the form $\Lambda[M]$ for a quasitilted algebra Λ and a Λ -module M . We will also need from [HRS2] that in this case the indecomposable direct summands of M are contained in $\mathcal{L}$. Moreover the following results are established in [HRS2], Lemmas 2.1 and 2.2.

Lemma 1.3. *Let Λ be a k -algebra with $\text{gl.dim } \Lambda \leq 2$ and let $\Delta = \Lambda[M]$ for a Λ -module M . Let ${}_\Delta Y = (k^t, {}_\Lambda X, f)$ be in $\text{mod } \Delta$. Then*

- (i) *If $\ker f$ is not projective, then $\text{pd}_\Delta Y \geq 2$.*
- (ii) *Assume that $\text{pd}_\Lambda \text{coker } f \leq 1$. Then $\text{pd}_\Delta Y \leq 1$ if and only if $\ker f$ is projective.*
- (iii) *$\text{id}_\Delta Y \leq 1$ if and only if $\text{id}_\Lambda X \leq 1$ and $\text{Ext}_\Lambda^1(M, X) = 0$.*

The following observations will be useful in the next section.

Lemma 1.4. *Let Λ be a k -algebra and let $\Delta = \Lambda[M]$ for a Λ -module $M = M_1 \oplus M_2$ with $M_1 \neq 0 \neq M_2$. Let X_1, X_2 be two indecomposable nonisomorphic Λ -modules and $f_i : M_i \rightarrow X_i$ be non-zero maps for $i = 1, 2$. Let ${}_\Delta Y = (k, {}_\Lambda(X_1 \oplus X_2), f = \begin{pmatrix} f_1 & 0 \\ 0 & f_2 \end{pmatrix})$ be in $\text{mod } \Delta$. Then Y is indecomposable.*

Proof. Indeed, if Y is decomposable then there exists i such that $(0, X_i, 0)$ is a direct summand of Y . We may assume that $i = 1$. This gives rise to the following

commutative diagram of Λ -modules:

$$\begin{array}{ccc} 0 & \longrightarrow & X_1 \\ \downarrow & & \downarrow \alpha \\ M_1 \oplus M_2 & \xrightarrow{f} & X_1 \oplus X_2 \\ \downarrow & & \downarrow \beta \\ 0 & \longrightarrow & X_1 \end{array}$$

with $f\beta = 0$ and $\alpha\beta = 1_{X_1}$. Writing $\alpha = (\alpha_1, \alpha_2)$ and $\beta = \begin{pmatrix} \beta_1 \\ \beta_2 \end{pmatrix}$ we obtain $f_1\beta_1 = 0 = f_2\beta_2$ and $\alpha_1\beta_1 + \alpha_2\beta_2 = 1_{X_1}$. Since X_1 is indecomposable and $X_1 \not\cong X_2$ we infer that $\alpha_2\beta_2$ is nilpotent, thus $\alpha_1\beta_1 = 1_{X_1} - \alpha_2\beta_2$ is invertible. In particular β_1 is invertible, and therefore $f_1 = 0$, contrary to our assumption. Thus Y is an indecomposable Δ -module. $\square$

Lemma 1.5. *Let Λ be a k -algebra and let $\Delta = \Lambda[M]$ for a Λ -module $M = M_1 \oplus M_2$ with $M_1 \neq 0 \neq M_2$. Let X be an indecomposable Λ -module and $f_i : M_i \rightarrow X$ be maps for $i = 1, 2$ which are not both equal to zero. Let ${}_{\Delta}Y = (k, {}_{\Lambda}X, f = \begin{pmatrix} f_1 \\ f_2 \end{pmatrix})$ be in $\text{mod } \Delta$. Then Y is indecomposable.*

Proof. Indeed, if Y is decomposable then we have that $(0, X, 0)$ is a direct summand of Y . This gives rise to following commutative diagram of Λ -modules:

$$\begin{array}{ccc} 0 & \longrightarrow & X \\ \downarrow & & \downarrow \alpha \\ M_1 \oplus M_2 & \xrightarrow{f} & X \\ \downarrow & & \downarrow \beta \\ 0 & \longrightarrow & X \end{array}$$

where α and β are isomorphisms and $f_1\beta = f_2\beta = 0$. Thus $f_1 = f_2 = 0$ contrary to our assumption. Thus Y is an indecomposable Δ -module. $\square$

In the proof of the main result we will make use of the criterion in [DP]. For the convenience of the reader we will recall this result. Before doing so, we have to introduce some further notation. Recall that we may define for a finite-dimensional k -algebra Λ the *quiver* $Q(\Lambda)$. The vertices of $Q(\Lambda)$ are the isomorphism classes $[S]$ of simple Λ -modules S , and the number of arrows from $[S']$ to $[S]$ is the dimension of $\text{Ext}_{\Lambda}^1(S, S')$. We also consider a partial order on the vertices of $Q(\Lambda)$ by defining $a \preceq b$ if there is a path in $Q(\Lambda)$ from a to b . Note that this implies that there is a path in $\text{ind } \Lambda$ from $P(a)$ to $P(b)$, where for a vertex $c \in Q(\Lambda)$ we have denoted by $P(c)$ the projective cover of the simple Λ -module $S(c)$ corresponding to the vertex c . Given any Λ -module N , we define the *support algebra* of N as the factor algebra of Λ modulo the ideal generated by all idempotents which annihilate N . Given a vertex $a \in Q(\Lambda)$, we define Λ^a as the support algebra of $\bigoplus_{a \not\preceq b} S(b)$. If $Q(\Lambda)$ has no oriented cycle we infer that the radical $\text{rad } P(a)$ of $P(a)$ is an Λ^a -module. Given $a \in Q(\Lambda)$, we denote by $\text{rad } P(a) = \bigoplus_{i=1}^{n_a} R_i(a)$ the decomposition of $\text{rad } P(a)$ into

indecomposable direct summands. Using this notation we have the following result from [DP].

Theorem 1.6. *Let Λ be a finite-dimensional algebra whose quiver $Q(\Lambda)$ has no oriented cycle. Then the Auslander-Reiten quiver Γ_Λ has a preprojective component if and only if for every vertex $a \in Q(\Lambda)$ one of the following conditions is satisfied:*

- (i) *There is a preprojective component $\mathcal{P}$ of Γ_{Λ^a} such that $R_i(a) \notin \mathcal{P}$ for every $1 \leq i \leq n_a$.*
- (ii) *For each $1 \leq i \leq n_a$ the set of predecessors $\{Y \in \Gamma_{\Lambda^a} \mid Y \preceq R_i(a)\}$ of $R_i(a)$ in $\text{ind } \Lambda^a$ is finite and consists of directing modules. If a is a sink, then $\text{rad } P(a)$ is a directing module in $\text{mod } \Lambda^a$.*

2. THE MAIN RESULT

We keep the notation from the previous sections.

Lemma 2.1. *Let Λ be a quasitilted algebra and $M = M_1 \oplus M_2$ a Λ -module such that $\Delta = \Lambda[M]$ is a quasitilted algebra. Then each indecomposable direct summand of M_1 is contained in $\mathcal{R}(\Lambda)$ or M_2 is projective.*

Proof. Suppose that there exists an indecomposable direct summand M'_1 of M_1 with $M'_1 \notin \mathcal{R}(\Lambda)$ and that M_2 is not projective. Consider the Δ -module $Y = (k, M'_1, \pi'_1)$ where π'_1 is the projection onto M'_1 . By Lemma 1.5 we have that Y is indecomposable and by 1.3 we have that $\text{pd}_\Delta Y = 2$. Thus there exists an indecomposable injective Δ -module ${}_\Delta I$ such that $\text{Hom}_\Delta(I, \tau_\Delta Y) \neq 0$. Therefore there exists a path $(I, \tau Y, E, Y)$ in $\text{ind } \Delta$, where E is an indecomposable direct summand of the middle term of the Auslander-Reiten sequence ending in Y . Since $M'_1 \notin \mathcal{R}(\Lambda)$, there is a path in $\text{ind } \Lambda$ from M'_1 to an indecomposable Λ -module X with $\text{id}_\Lambda X = 2$. In particular, $X \in \mathcal{L}(\Lambda)$. By Lemma 1.2 there is a path $M'_1 \xrightarrow{f} F \xrightarrow{g} X$ in $\text{ind } \Lambda$. If $fg \neq 0$, then, by Lemma 1.3, the indecomposable Δ -module $(k, X, \begin{pmatrix} \hat{f}g \\ 0 \end{pmatrix})$ has both projective dimension and injective dimension equal to two, a contradiction. Thus $fg = 0$. Since $\text{id}_\Lambda X = 2$, we infer that $\text{Hom}_\Lambda(\tau_\Lambda^- X, {}_\Lambda P) \neq 0$ for some indecomposable projective Λ -module P . The following commutative diagram of Λ -modules shows that there exists a path in $\text{ind } \Delta$ from Y to $(0, X, 0)$:

$$\begin{array}{ccc} M_1 \oplus M_2 & \xrightarrow{\pi'_1} & M'_1 \\ \downarrow & & \downarrow f \\ M_1 \oplus M_2 & \xrightarrow{\hat{f}} & F \\ \downarrow & & \downarrow g \\ 0 & \longrightarrow & X \end{array}$$

where $\hat{f}$ is the extended map as defined in the introduction. We thus obtain by combining the constructed paths a non-sectional path in $\text{ind } \Delta$ from I to ${}_\Lambda P$. Since P is also Δ -projective, we have a contradiction to 1.1. $\square$

We also point out the following easy consequence.

Corollary 2.2. *Let Λ be a quasitilted algebra and $M = \oplus_i^t M_i$ a Λ -module with $t \geq 2$ such that $\Delta = \Lambda[M]$ is a quasitilted algebra. If $M_i \notin \mathcal{R}(\Lambda)$ for all $1 \leq i \leq t$, then M is projective.*

Theorem 2.3. *Let Δ be a quasitilted algebra. Then the Auslander-Reiten quiver of Δ has a preprojective component.*

Proof. We may assume that Δ is not tilted. In this case $\mathcal{L} \setminus \mathcal{R}$ is a union of components of Γ_Δ by [CS]. In particular these components do not contain any indecomposable injective Δ -module.

We will prove the theorem by induction on the number n of simple Δ -modules. For $n = 1$ there is nothing to show. So assume that all quasitilted algebras with less than n simple modules have a preprojective component.

Let Δ be a quasitilted algebra with n simple modules. Let $Q(\Delta)$ be the quiver of Δ . Let $a \in Q(\Delta)$ be a vertex. If a is not a sink, then there exist a sink ω and a path from a to ω . Let $M = \text{rad } P(\omega)$. Then there exists a quasitilted algebra Λ such that $\Delta = \Lambda[M]$ and also $\Delta^a = \Lambda^a$. By induction, Λ has a preprojective component, so the vertex a satisfies one of the conditions in 1.6. Thus we are left with the case that $a = \omega$ is a sink. As noted before, we can write $\Delta = \Lambda[M]$ for a quasitilted algebra Λ and a Λ -module $M = \text{rad } P(\omega)$. By induction we have that the Auslander-Reiten quiver of Λ has a preprojective component. We will show that ω satisfies one of the conditions of 1.6.

For this let M_1 be the direct sum of those indecomposable direct summands of M which are contained in the preprojective components of Γ_Λ . Then $M = M_1 \oplus M_2$. If $\mathcal{P}$ is a preprojective component of Γ_Λ we may assume that $\mathcal{P}$ contains an indecomposable summand of M_1 . Otherwise, $\mathcal{P}$ will also be a preprojective component for the Auslander-Reiten quiver of Δ ; compare [DP]. In particular we will assume from now on that $M_1 \neq 0$ and that ω does not satisfy condition (i) of 1.6.

We will show first that M_2 is projective.

If M_2 is not projective, there exists an indecomposable non-projective direct summand M'_2 of M_2 . Let Λ_1 be the connected component of Λ supporting M'_2 . We first consider the case that all indecomposable projective Λ_1 -modules are contained in preprojective components of Γ_{Λ_1} . Let P be an indecomposable projective Λ_1 -module with $\text{Hom}_{\Lambda_1}(P, \tau_{\Lambda_1} M'_2) \neq 0$. By assumption we have that P is contained in a preprojective component $\mathcal{P}$ which also contains an indecomposable direct summand M'_1 of M_1 . Since $\text{Hom}_{\Lambda_1}(P, \tau_{\Lambda_1} M'_2) \neq 0$ and $M'_2 \notin \mathcal{P}$, there exists $X \in \mathcal{P}$ with $M'_1 \preceq X$ and $\text{Hom}_{\Lambda_1}(X, \tau_{\Lambda_1} M'_2) \neq 0$. In particular we obtain a path from M'_1 to $\tau_{\Lambda_1} M'_2$. By Lemma 1.2 there exist an indecomposable Λ_1 -module Y and a path $M'_1 \xrightarrow{f} Y \xrightarrow{g} \tau M'_2$ in $\text{ind } \Lambda_1$. We consider the Δ -module $Z = (k, Y, \alpha = \begin{pmatrix} f \\ 0 \end{pmatrix})$.

Then Z is indecomposable and by Lemma 1.3 we infer that $\text{pd}_\Delta Z = 2 = \text{id}_\Delta Z$. For this note that M'_2 is a direct summand of $\ker \alpha$ and $\text{Ext}_{\Lambda_1}^1(M, Y) \neq 0$, for $\text{Hom}_{\Lambda_1}(Y, \tau M) \neq 0$. This contradicts the fact that Δ is quasitilted. Therefore there exists an indecomposable projective Λ_1 -module which is not contained in a preprojective component of Γ_{Λ_1} . Since Λ_1 is connected, there exist indecomposable projective Λ_1 -modules P, P' such that $\text{Hom}_{\Lambda_1}(P, P') \neq 0$ and P is contained in a preprojective component $\mathcal{P}$ of Γ_{Λ_1} and P' is not contained in a preprojective component of Γ_{Λ_1} . Again there is an indecomposable direct summand M'_1 of M_1 contained in $\mathcal{P}$. Since M_2 is not projective, we have by Lemma 2.1 that $M'_1 \in$

$\mathcal{R}(\Lambda)$. Let $0 \neq f \in \text{Hom}_{\Lambda_1}(P, P')$. Then by the choice of P, P' we have that $f \in \text{rad}^\infty(P, P')$. Thus for each $r \geq 1$, there exist a chain of irreducible maps

$$P = X_0 \xrightarrow{f_1} X_1 \xrightarrow{f_2} \dots \xrightarrow{f_r} X_r$$

and a map $g_r : X_r \rightarrow P'$ such that $f_1 \dots f_r g_r \neq 0$. Choose r in such a way that τX_r is a successor of M'_1 . Note that $\text{id}_\Lambda \tau X_r = 2$. Since $\mathcal{R}(\Lambda)$ is closed under successors and $M'_1 \in \mathcal{R}(\Lambda)$ we infer that $\tau X_r \in \mathcal{R}(\Lambda)$, a contradiction. Thus M_2 is projective.

Assume that $M_2 \neq 0$. We will show next that in this case there exists an indecomposable Λ -module X with $\text{id}_\Lambda X = 2$ and $\text{Hom}_\Lambda(M_1, X) \neq 0$.

By the previous part of the proof we know that M_2 is projective. Let M'_2 be an indecomposable direct summand of M_2 and as before let Λ_1 be the connected component of Λ supporting M'_2 . Let M'_1 be an indecomposable direct summand of M_1 . Then M'_1 lies in a preprojective component $\mathcal{P}$ of Γ_{Λ_1} . We consider $\mathcal{S}(M'_1 \rightarrow)$, the subset of $\mathcal{P}$ consisting of those indecomposable modules X for which there is a sectional path from M'_1 to X .

We distinguish the following two cases.

First assume that there is no proper successor of $\mathcal{S}(M'_1 \rightarrow)$ which is projective. Arguing as above we find $X \in \mathcal{S}(M'_1 \rightarrow)$ with $\text{id}_\Lambda X = 2$. Indeed, since Λ_1 is connected there exist indecomposable projective Λ_1 -modules P, P' such that $\text{Hom}_{\Lambda_1}(P, P') \neq 0$ and P is contained in $\mathcal{P}$ and P' is not contained in a preprojective component of Γ_{Λ_1} . By the choice of P, P' we have that $f \in \text{rad}^\infty(P, P')$. Thus for each $r \geq 1$, there exists a chain of irreducible maps

$$P = X_0 \xrightarrow{f_1} X_1 \xrightarrow{f_2} \dots \xrightarrow{f_r} X_r$$

and a map $g_r : X_r \rightarrow P'$ such that $f_1 \dots f_r g_r \neq 0$. Choose r in such a way that τX_r is contained in $\mathcal{S}(M'_1 \rightarrow)$. Note that $\text{id}_\Lambda \tau X_r = 2$.

Next assume that there is a proper successor $\mathcal{S}(M'_1 \rightarrow)$ which is projective. Consider $\Sigma = \mathcal{S}(\rightarrow P)$, the subset of $\mathcal{P}$ consisting of those indecomposable modules X for which there is a sectional path from X to P . Note that the indecomposable modules in $\tau\Sigma$ all have injective dimension two and that there is a path from M'_1 to an indecomposable module in $\tau\Sigma$. Also note that $\tau\Sigma$ is a separating subcategory in the sense that each map from a predecessor of $\tau\Sigma$ in $\mathcal{P}$ to a module which is not a predecessor of $\tau\Sigma$ factors through $\tau\Sigma$. We consider a nonzero map M'_1 to an indecomposable injective module. Since any path from an indecomposable injective to an indecomposable projective is sectional for a quasitilted algebra, we infer that I is not a predecessor of $\tau\Sigma$. Hence there is a nonzero map from M'_1 to an indecomposable module in $\tau\Sigma$. This proves our claim.

Next we will show that $\text{Hom}_\Lambda(M_2, Y) = 0$ for all $Y \in \text{ind } \Lambda$ with $\text{pd}_\Lambda Y = 2$.

Suppose to the contrary that there exists an indecomposable Λ -module Y with $\text{pd}_\Lambda Y = 2$ and $\text{Hom}_\Lambda(M_2, Y) \neq 0$. By the previous claim there also exists an indecomposable Λ -module X with $\text{id}_\Lambda X = 2$ and $\text{Hom}_\Lambda(M_1, X) \neq 0$. Choose nonzero maps $f : M_1 \rightarrow X$ and $g : M_2 \rightarrow Y$. Consider the Δ -module

$$Z = \left(k, X \oplus Y, \begin{pmatrix} f & 0 \\ 0 & g \end{pmatrix} \right).$$

By Lemma 1.4 we have that Z is indecomposable and by Lemma 1.3 we have that $\text{pd}_\Delta Z = 2 = \text{id}_\Delta Z$, a contradiction.

It follows from this that each submodule of M_2 is projective, since otherwise the corresponding factor module would have projective dimension two. Moreover

it follows that M_2 is directing and each indecomposable summand of M_2 has only finitely many predecessors. In fact let X be an indecomposable Λ -module with $0 \neq f \in \text{Hom}(X, M'_2)$ for an indecomposable direct summand M'_2 of M_2 . Then let $f = \pi\mu$ be the canonical factorization through $B = \text{im } f$. So B is projective, and hence X is projective and isomorphic to B . As a consequence of this we infer that there is no path from an indecomposable direct summand of M_1 to an indecomposable direct summand of M_2 .

As a final step we will show that M is directing as a Λ -module. By the previous remark it is enough to show that M_1 is directing. Suppose that there exist indecomposable direct summands M'_1 and M''_1 of M_1 and a non-sectional path from M'_1 to M''_1 .

If M''_1 is not projective there exists a path from M'_1 to $\tau M''_1$. By Lemma 1.2 there exists a path $M'_1 \xrightarrow{f} Y \xrightarrow{g} \tau M''_1$. Consider the indecomposable Δ -module $Z = (k, Y, \begin{pmatrix} \hat{f} \\ 0 \end{pmatrix})$, which again by Lemma 1.3 has both projective and injective dimension two. So we have that M''_1 is projective. Again by Lemma 1.2 we obtain a path $M'_1 \xrightarrow{f} Y \xrightarrow{g} M''_1$. We will show that there exists an indecomposable non-projective Λ -module W such that $\text{Hom}_\Lambda(M'_1, \tau W) \neq 0$ and $\text{Hom}_\Lambda(W, M''_1) \neq 0$. If $fg = 0$ this follows from [HR]. So suppose that $fg \neq 0$ and let $\Sigma = \mathcal{S}(\rightarrow M''_1)$. By assumption we have that M'_1 is a predecessor of $\tau\Sigma$. Therefore the nonzero map fg factors through a module in $\tau\Sigma$. In particular there exists an indecomposable non-projective Λ -module W such that $\text{Hom}_\Lambda(M'_1, \tau W) \neq 0$ and $\text{Hom}_\Lambda(W, M''_1) \neq 0$. Since M''_1 is projective we infer that $\text{id}_\Lambda \tau W = 2$. Let $0 \neq \alpha \in \text{Hom}_\Lambda(M'_1, \tau W)$ and $0 \neq \beta \in \text{Hom}_\Lambda(W, M''_1)$. Since W is non-projective we have that β is not surjective. Also, $\text{pd}_\Lambda \text{coker } \beta = 2$. Thus there exist an indecomposable Λ -module Y with $\text{pd}_\Lambda Y = 2$ and a nonzero map $\gamma : M''_1 \rightarrow Y$. Consider the indecomposable Δ -module $Z = (k, \tau W \oplus Y, \begin{pmatrix} \hat{\alpha} & 0 \\ 0 & \hat{\gamma} \end{pmatrix})$, which again by Lemma 1.3 has both projective and injective dimension two, a contradiction. Thus M_1 , and therefore M , is directing.

This shows that the extension vertex $\omega \in Q(\Delta)$ satisfies the condition (ii) of Theorem 1.6. In fact, we have just seen that $M = \text{rad } P(\omega)$ is directing. Also any indecomposable summand M_2 has only finitely predecessors, all of which are directing. The indecomposable direct summands of M_1 are all contained in preprojective components of Γ_Λ , and hence there are only finitely many predecessors and all are directing. Note that $\Lambda = \Delta^\omega$.

This finishes the proof of the theorem. $\square$

3. EXAMPLE

In this section we will consider the following example. Let Λ be given by the path algebra of the following quiver modulo the ideal generated by all paths of length 2:

The Auslander-Reiten quiver of Λ is given as follows. This may be used to verify the remarks below.

Let S be the simple projective in the unique preprojective component and let S' be the simple projective in the preinjective component. Let M be the sum of the two simple projective Λ -modules. We consider $\Delta = \Lambda[M]$. We claim that Δ is a quasitilted algebra. In fact, let $Z = (k^t, X, f)$ be an indecomposable Δ -module. We may assume that $t \geq 1$; compare [HRS2]. It is easily seen that $\text{Hom}_\Delta(M, Y) = 0 = \text{Ext}_\Delta^1(Y, M)$ for all indecomposable Λ -modules Y with $\text{pd}_\Lambda Y = 2$. Since Z is indecomposable, it follows from Lemma 1.3 that $\text{pd}_\Delta Z \leq 1$. Thus Δ is a quasitilted algebra, and hence has a preprojective component. This of course may be verified by a direct computation. Note that Δ is a tilted algebra, for the preinjective component contains a complete slice.

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