A CONVOLUTION ESTIMATE FOR A MEASURE ON A CURVE IN $\mathbb{R}^4$

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ABSTRACT. Let $\gamma(t) = (t, t^2, t^3, t^4)$ and fix an interval $I \subset \mathbb{R}$. If T is the operator on $\mathbb{R}^4$ defined by $Tf(x) = \int_I f(x - \gamma(t)) dt$, then T maps $L^{\frac{5}{3}}(\mathbb{R}^4)$ into $L^2(\mathbb{R}^4)$.

INTRODUCTION

For $n \geq 2$, let $\gamma(t) = (t, t^2, \dots, t^n)$ for $t \in \mathbb{R}$. For an interval $I \subseteq \mathbb{R}$, define the operator T by

$$Tf(x) = \int_I f(x - \gamma(t)) dt$$

for suitable functions f on $\mathbb{R}^n$. We are interested in determining the type set $\mathcal{T}$ of T — the set of points $(\frac{1}{p}, \frac{1}{q}) \in [0, 1] \times [0, 1]$ such that T maps $L^p(\mathbb{R}^n)$ into $L^q(\mathbb{R}^n)$. Let S be the closed segment, degenerate if $n = 2$, on the line $\frac{1}{p} - \frac{1}{q} = \frac{2}{n(n+1)}$ with endpoints $(\frac{n^2-n+2}{n^2+n}, \frac{n-1}{n+1})$ and $(\frac{2}{n+1}, \frac{2n-2}{n^2+n})$. Then (see §4 of [4]) $\mathcal{T}$ is contained in the closed convex hull of S and $\{(0, 0), (1, 1)\}$ if I is bounded, and in S if I is unbounded. If $n = 2$ or 3 these inclusions are equalities. We conjecture that this is true for any n . The main result in the case $n = 3$ is that T maps $L^{\frac{3}{2}}(\mathbb{R}^3)$ into $L^2(\mathbb{R}^3)$. This was proved in [5]. The method there was later used in [2], [8], [9], and [10] to treat more general curves in $\mathbb{R}^3$. The only nontrivial result for $n > 3$ is the fact, proved in [4], that the midpoint of S lies in $\mathcal{T}$. The purpose of this paper is to prove a partial result if $n = 4$.

Theorem 1. *Suppose $n = 4$. The segment on $\frac{1}{p} - \frac{1}{q} = \frac{1}{10}$ with endpoints $(\frac{1}{2}, \frac{2}{5})$ and $(\frac{3}{5}, \frac{1}{2})$ lies in $\mathcal{T}$.*

Our approach here is different from those employed previously but, relying on the method of T^*T , it suffers from the same flaw as the method of [5]: the range space must be L^2 .

Let P be the polynomial on $\mathbb{R}^4$ defined by

$$P(a, b, c, d) = 108a^2d^2 + 32b^3d - 108abcd + 27ac^3 - 9c^2b^2.$$

Theorem 2. *The function $|P|^{-\frac{1}{10}}$ is a Fourier multiplier of $L^{\frac{5}{3}}(\mathbb{R}^4)$ into $L^2(\mathbb{R}^4)$.*

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Theorem 3. *There is an absolute constant C such that*

$$\left| \int_I e^{i(at+bt^2+ct^3+dt^4)} dt \right| \leq C |P(a, b, c, d)|^{-\frac{1}{10}}$$

for any $(a, b, c, d) \in \mathbb{R}^4$ and any interval $I \subseteq \mathbb{R}$.

Theorem 1 follows from Theorems 2 and 3 since

$$\widehat{Tf}(a, b, c, d) = \widehat{f}(a, b, c, d) \int_I e^{-i(at+bt^2+ct^3+dt^4)} dt.$$

(The importance of L^2 as range space here is evident.) There are simpler analogues of Theorems 2 and 3 that yield a proof of the $L^{\frac{3}{2}}(\mathbb{R}^3) - L^2(\mathbb{R}^3)$ result as well: let

$$Q(a, b, c) = b^2 - 3ac.$$

Theorem 2'. *The function $|Q|^{-\frac{1}{4}}$ is a Fourier multiplier of $L^{\frac{3}{2}}(\mathbb{R}^3)$ into $L^2(\mathbb{R}^3)$.*

Theorem 3'. *There is an absolute constant C such that*

$$\left| \int_I e^{i(at+bt^2+ct^3)} dt \right| \leq C |Q(a, b, c)|^{-\frac{1}{4}}.$$

It seems likely that there are also analogues of Theorems 2 and 3 for $n \geq 5$ and that these will yield an analogue of Theorem 1: boundedness of T on the segment on $\frac{1}{p} - \frac{1}{q} = \frac{2}{n(n+1)}$ with endpoints $(\frac{1}{2}, \frac{1}{q})$ and $(\frac{1}{p}, \frac{1}{2})$. But there are some significant technical complications as n increases.

Theorems 3 and 3' are proved in [7], which contains estimates for integrals

$$\int_I e^{ip(t)} dt$$

in case p is a polynomial with real coefficients. Most of this paper is devoted to the proof of Theorem 2. But we begin with a sketch of the (quite routine) proof of Theorem 2', since it is based on the two principles, the method of T^*T and analytic interpolation, which yield Theorem 2. In what follows, the positive constant C may increase from line to line but will depend only on the parameters n , p , and q .

We sketch the proof of Theorem 2'. Suppose $1 \leq p \leq 2 \leq q \leq \infty$ and $\frac{1}{p} + \frac{1}{q} = 1$. The "method of T^*T " is the statement that an operator T is bounded from L^p to L^2 if and only if T^*T is bounded from L^p to L^q . The proof is that both statements are equivalent to

$$|\langle T^*Tf, g \rangle| = |\langle Tf, Tg \rangle| \leq C \|f\|_p \|g\|_p.$$

Thus Theorem 2' is equivalent to the statement that $|Q|^{-\frac{1}{2}}$ is a Fourier multiplier of $L^{\frac{3}{2}}(\mathbb{R}^3)$ into $L^3(\mathbb{R}^3)$. By a linear change of variables we can assume that $Q(a, b, c) = a^2 + b^2 - c^2$. Consider the analytic family of multipliers

$$m_z = (z+1) \left(z + \frac{3}{2} \right) |Q|^z, \quad -\frac{3}{2} \leq \operatorname{Re} z \leq 0.$$

If $\operatorname{Re} z = 0$, these are bounded on $L^2(\mathbb{R}^3)$ with the multiplier norm growing polynomially in $\operatorname{Im} z$. If $\operatorname{Re} z = -\frac{3}{2}$, the Fourier transform formulas of [3] show that the L^∞ norms of $\widehat{m}_z$ (and so the $L^1(\mathbb{R}^3)$ to $L^\infty(\mathbb{R}^3)$ multiplier norms of m_z) grow

polynomially in $\operatorname{Im} z$. Thus Stein's theorem on analytic interpolation shows that $|Q|^{-\frac{1}{2}}$ is a Fourier multiplier of $L^{\frac{3}{2}}(\mathbb{R}^3)$ into $L^3(\mathbb{R}^3)$.

PROOF OF THEOREM 2

We begin with a localization lemma, given in more generality than we really require. With a positive integer n fixed, let $M(p, q)$ denote the collection of Fourier multipliers from $L^p(\mathbb{R}^n)$ into $L^q(\mathbb{R}^n)$. We will say that a function m on $\mathbb{R}^n$ belongs locally to $M(p, q)$ at $\xi \in \mathbb{R}^n$ if there is some function φ on $\mathbb{R}^n$ with $\varphi = 1$ in a neighborhood of ξ such that $m\varphi \in M(p, q)$.

Lemma. *Suppose $1 < p \leq 2 \leq q < \infty$ and m is homogeneous of degree $\frac{n}{q} - \frac{n}{p}$. If m belongs locally to $M(p, q)$ at each nonzero $\xi \in \mathbb{R}^n$, then $m \in M(p, q)$.*

The proof is an easy consequence of the Littlewood-Paley decomposition: Suppose Δ is the collection of rectangles effecting the standard dyadic decomposition of $\mathbb{R}^n$. If $\rho \in \Delta$, let S_ρ be the multiplier operator with symbol χ_ρ . Then the two estimates

$$\left(\sum_{\rho \in \Delta} \|S_\rho f\|_p^2 \right)^{\frac{1}{2}} \leq C \|f\|_p,$$

$$\|f\|_q \leq C \left(\sum_{\rho \in \Delta} \|S_\rho f\|_q^2 \right)^{\frac{1}{2}}$$

are well-known consequences of the Littlewood-Paley decomposition and Minkowski's inequality. Together they show that if S is the multiplier operator with symbol m , then the lemma will follow from the estimates, uniform in ρ ,

$$(2.1) \quad \|SS_\rho f\|_q \leq C \|S_\rho f\|_p.$$

(See [1] for an earlier application of this idea.)

It is a consequence of the locality hypothesis on m and a simple argument involving compactness and a partition of unity that there is a function φ on $\mathbb{R}^n$ satisfying

$$m\varphi \in M(p, q), \quad \varphi(\xi) = 1 \quad \text{if } 1 \leq |\xi| \leq \sqrt{n}.$$

Fix $\rho \in \Delta$ and let 2^j be the length of the longest side of ρ . Then $\varphi(\xi) = 1$ on the support of $\chi_\rho(2^j \xi)$. Letting $\|\cdot\|_{pq}$ denote the multiplier norm, it follows from $m\varphi \in M(p, q)$ that

$$(2.2) \quad \|m(\xi)\chi_\rho(2^j \xi)\|_{pq} \leq C.$$

A homogeneity argument shows that

$$\|m(2^{-j} \xi)\chi_\rho(\xi)\|_{pq} = 2^{-j(\frac{n}{q} - \frac{n}{p})} \|m(\xi)\chi_\rho(2^j \xi)\|_{pq}.$$

Now (2.1) follows from (2.2) and the homogeneity hypothesis on m .

To apply the lemma to the proof of Theorem 2 we will take $n = 4$, $p = 2$, $q = \frac{5}{2}$, and $m = |P|^{-\frac{1}{10}}$. We must show that if $\xi_0 = (a_0, b_0, c_0, d_0) \neq 0$, then m belongs locally to $M(2, \frac{5}{2})$ at ξ_0 . There are three cases: $P(\xi_0) \neq 0$, $P(\xi_0) = 0$ and $\operatorname{grad} P(\xi_0) \neq 0$, $P(\xi_0) = 0$ and $\operatorname{grad} P(\xi_0) = 0$. The first case is easy — if $P(\xi_0) \neq 0$, then m is bounded on some neighborhood of ξ_0 . And any measurable

function bounded in a neighborhood of ξ_0 belongs locally to $M(2, q)$ at ξ_0 for any $q \geq 2$.

The second and third cases follow the pattern of the proof of Theorem 2': roughly, if N is an appropriate neighborhood of ξ_0 , we consider an analytic family of multipliers defined formally by

$$m_z(\xi) = p(z)|P(\xi)|^z \chi_N(\xi)$$

for some polynomial p . If $\operatorname{Re} z = 0$ (respectively, $\operatorname{Re} z = -1$) the multiplier norms $\|m_z\|_{22}$ (respectively, $\|m_z\|_{1\infty}$) turn out to have polynomial growth in $|\operatorname{Im} z|$. So $m_{-\frac{2}{10}}$ is in $M(\frac{5}{3}, \frac{5}{2})$ by Stein's theorem on interpolation with analytic families. Thus

$$m\chi_N \in M\left(2, \frac{5}{2}\right)$$

by the method of T^*T . The details of this argument are routine for the second case, more complicated in the third.

Let Σ be the surface in $\mathbb{R}^4$ defined by $P = 0$, and write $d\sigma$ for surface area measure on Σ . In the second case ($P(\xi_0) = 0$, $\operatorname{grad} P(\xi_0) \neq 0$) choose a small neighborhood $\tilde{N}$ of ξ_0 in Σ and a small positive number δ such that the mapping

$$\xi = \sigma + t \operatorname{grad} P(\sigma), \quad \sigma \in \tilde{N}, \quad t \in (-\delta, \delta),$$

gives a one-to-one parametrization of some neighborhood N of ξ_0 on which

$$\frac{d\xi}{C} \leq d\sigma \, dt \leq C \, d\xi,$$

$$(2.3) \quad \frac{1}{C} \leq |\operatorname{grad} P(\sigma)|, \quad \text{and}$$

$$\frac{|t|}{C} \leq |P(\xi)|.$$

Consider the family of operators defined, for $x \in \mathbb{R}^4$, by

$$T_z f(x) = (z+1) \int_{\tilde{N}} \int_{-\delta}^{\delta} \hat{f}(\sigma + t \operatorname{grad} P(\sigma)) e^{ix \cdot (\sigma + t \operatorname{grad} P(\sigma))} |t|^z \, dt \, d\sigma.$$

From the conditions (2.3) it follows that we need only check that the quantities defined formally by

$$\sup_{x \in \mathbb{R}^4} \left| (z+1) \int_{\tilde{N}} \int_{-\delta}^{\delta} e^{ix \cdot (\sigma + t \operatorname{grad} P(\sigma))} |t|^z \, dt \, d\sigma \right|$$

have polynomial growth in $\operatorname{Im} z$ if $\operatorname{Re} z = -1$. Since

$$\int_{\tilde{N}} d\sigma < \infty$$

($\tilde{N}$ is small), this follows from

$$\sup_{u \in \mathbb{R}} \left| \int_0^{\delta} u e^{iut} t^{is} \, dt \right| \leq C(1 + |s|),$$

which is a consequence of van der Corput's lemma.

Consider now the third case: $\xi_0 = (a_0, b_0, c_0, d_0) \neq 0$, $P(\xi_0) = 0$, $\text{grad } P(\xi_0) = 0$. It is easy to check that not both a_0 and d_0 can be 0. We will assume $a_0 \neq 0$, the other case being similar. Let $N = \{\xi : |\xi - \xi_0| < \frac{|a_0|}{2}\}$, so that $|a| \geq \frac{|a_0|}{2}$ if $(a, b, c, d) \in N$. The equation $P(a, b, c, d) = 0$ is a quadratic equation in d , and solving it for d yields functions d_+ and d_- defined by

$$d_{\pm}(a, b, c) = \frac{27abc - 8b^3 \pm (4b^2 - 9ac)^{\frac{3}{2}}}{54a^2}.$$

Additionally, put

$$d_0(a, b, c) = \frac{27abc - 8b^3}{54a^2}.$$

Define $N_1 = N \cap \{9ac - 4b^2 < 0\}$ and $N_2 = N \cap \{9ac - 4b^2 > 0\}$, and let B_1 (respectively, B_2) be the projection of N_1 (respectively, N_2) onto $\{c = d = 0\}$. We start by recording some estimates for $|P|$ on N_1 and N_2 .

A little algebra shows that if $a \neq 0$ then

$$P(a, b, c, d) = 108a^2(d - d_0)^2 + \frac{(9ac - 4b^2)^3}{27a^2}.$$

Since $\frac{|a_0|}{2} \leq |a| \leq C$ on N , it follows that

$$\frac{|d - d_0| \cdot |9ac - 4b^2|^{\frac{3}{2}}}{C} \leq |P(a, b, c, d)| \quad \text{if } (a, b, c, d) \in N_2.$$

If $(a, b, c, d) \in N_1$, then the estimates are

$$\frac{|d - d_+| \cdot |9ac - 4b^2|^{\frac{3}{2}}}{C} \leq |P(a, b, c, d)| \quad \text{if } d_0 \leq d,$$

$$\frac{|d - d_-| \cdot |9ac - 4b^2|^{\frac{3}{2}}}{C} \leq |P(a, b, c, d)| \quad \text{if } d \leq d_0.$$

For $j = -, 0, +$ define

$$\rho_j(a, b, c, d) = |d - d_j| \cdot |9ac - 4b^2|^{\frac{3}{2}}.$$

Then it is enough to show that

$$\rho_0^{-\frac{1}{10}} \chi_{B_2}, \quad \rho_+^{-\frac{1}{10}} \chi_{B_1}, \quad \text{and} \quad \rho_-^{-\frac{1}{10}} \chi_{B_1}$$

are in $M(2, \frac{5}{2})$. In accordance with our general plan (the method of T^*T combined with analytic interpolation) it is enough to show that the L^∞ norms of the inverse Fourier transforms of (the analytic continuations of)

$$(z+1) \left(z + \frac{2}{3}\right) \rho_0^z \chi_{B_2}, \quad (z+1) \left(z + \frac{2}{3}\right) \rho_+^z \chi_{B_1}, \quad (z+1) \left(z + \frac{2}{3}\right) \rho_-^z \chi_{B_1}$$

have polynomial growth in $|\text{Im } z|$ if $\text{Re } z = -1$. We will write s for $\text{Im } z$.

In the first case the quantity to estimate is, formally,

$$(2.4) \quad \int_{B_2} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} e^{i(x_1 a + x_2 b + x_3 c + x_4 d)} \frac{dd}{|d - d_0|^{1-is}} \frac{dc}{|9ac - 4b^2|^{3(1-is)/2}} d(b, a).$$

(The quantities above are defined by the analytic continuations of certain integrals. We maintain the integral notation only for convenience.)

Using the equation (see p. 359 of [3])

$$(2.5) \quad \int_{-\infty}^{\infty} e^{ix\sigma} |x|^\lambda dx = -2 \sin \frac{\lambda\pi}{2} \Gamma(\lambda+1) |\sigma|^{-\lambda-1} \quad (\lambda \neq -1, -3, \dots),$$

we see that it is enough to estimate

$$\int_{B_2} \int_{-\infty}^{\infty} e^{i(x_1 a + x_2 b + x_3 c + x_4 d_0)} \frac{dc}{|9ac - 4b^2|^{3(1-is)/2}} d(b, a).$$

Making the change of variable $C = 9ac - 4b^2$, we examine

$$\int_{B_2} \int_{-\infty}^{\infty} \exp \left\{ i \left[x_1 a + x_2 b + \frac{4x_3}{9a} b^2 + \frac{2x_4}{27a^2} b^3 + \left(\frac{x_3}{9a} + x_4 \frac{b}{18a^2} \right) C \right] \right\} \cdot \frac{dC}{|C|^{3(1-is)/2}} \frac{d(b, a)}{|a|}.$$

With $p_a(b) = x_1 a + x_2 b + \frac{4x_3}{9a} b^2 + \frac{2x_4}{27a^2} b^3$, use (2.5) again to obtain

$$\int_{B_2} \exp \{ i p_a(b) \} \left| \frac{p_a''(b)}{8} \right|^{\frac{1}{2}(1-3is)} db \frac{da}{|a|}.$$

For each a , the integral on b is over some interval and so, by [6], is bounded by $C(1 + |s|)^{\frac{1}{2}}$. The integral on a is over a compact interval not containing 0, and so the supremum over $(x_1, x_2, x_3, x_4) \in \mathbb{R}^4$ of the absolute value of (2.4) grows only polynomially in $|s|$.

The estimates for $(z+1)(z+\frac{2}{3})\rho_+^z \chi_{B_1}$ and $(z+1)(z+\frac{2}{3})\rho_-^z \chi_{B_1}$ are similar to each other but a little more complicated than that for $(z+1)(z+\frac{2}{3})\rho_0^z \chi_{B_2}$. It is enough to estimate

$$\int_{I_a} \int_{-\infty}^{\infty} e^{i(x_1 a + x_2 b + x_3 c + x_4 d_+)} \frac{dc}{|9ac - 4b^2|^{3(1-is)/2}} db,$$

where I_a is an interval depending on a . The change of variable $C = 9ac - 4b^2$ leads now to

$$(2.6) \quad \int_{I_a} \int_{-\infty}^{\infty} \exp \left\{ i \left[x_1 a + x_2 b + \frac{4x_3 b^2}{9a} + \frac{2x_4 b^3}{27a^2} + \left(\frac{x_3}{9a} + x_4 \frac{b}{18a^2} \right) C + \frac{x_4}{54a^2} C^{\frac{3}{2}} \right] \right\} \frac{dC}{|C|^{3(1-is)/2}} db.$$

Replacing b by $|x_4|^{-\frac{1}{3}} b$ and C by $|x_4|^{-\frac{2}{3}} C$ allows us to assume, without loss of generality, that $x_4 = 1$. The integration on b is over a new interval I'_a , and x_2 and x_3 must be replaced by new values x'_2 and x'_3 . Writing $p(b) = x_1 a + x'_2 b + \frac{4x'_3}{9a} b^2 + \frac{2}{27a^2} b^3$,

(2.6) becomes

$$(2.7) \quad \int_{I'_a} \int_{-\infty}^{\infty} \exp \left\{ i \left(p(b) + \frac{Cp''(b)}{8} + \frac{C^{\frac{3}{2}}}{54a^2} \right) \right\} \frac{dC}{|C|^{3(1-is)/2}} db.$$

The equality

$$\begin{aligned} & \exp \left\{ i \left(p(b) + \frac{Cp''(b)}{8} + \frac{C^{\frac{3}{2}}}{54a^2} \right) \right\} \\ &= \exp \left\{ i \left(p(b) + \frac{Cp''(b)}{8} \right) \right\} \left\{ 1 + \left[\exp \left(\frac{iC^{\frac{3}{2}}}{54a^2} \right) - 1 \right] \right\} \end{aligned}$$

splits (2.7) into the sum of two terms. We have already treated the first of these. The second can be written

$$(2.8) \quad \int_{-\infty}^{\infty} \int_{I'_a} \exp \left\{ i \left(p(b) + \frac{Cp''(b)}{8} \right) \right\} db \left[\exp \left(\frac{iC^{\frac{3}{2}}}{54a^2} \right) - 1 \right] \frac{dC}{|C|^{3(1-is)/2}}.$$

(The Fubini-like interchange here can be justified by uniqueness of analytic continuation.) The phase function $p(b) + \frac{Cp''(b)}{8}$ is a third degree polynomial with leading term $\frac{2b^3}{27a^2}$. Since a lies in a bounded interval, van der Corput's lemma bounds the absolute value of the inner term. Since a is bounded away from 0, (2.8) is bounded as well. This completes the proof of Theorem 2.

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