FIXED POINTS OF THE MAPPING CLASS GROUP IN THE $SU(n)$ MODULI SPACES

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ABSTRACT. Let Σ be a compact oriented surface with or without boundary components. In this note we prove that if $\chi(\Sigma) < 0$ then there exist infinitely many integers n such that there is a point in the moduli space of irreducible flat $SU(n)$ connections on Σ which is fixed by any orientation preserving diffeomorphism of Σ . Secondly we prove that for each orientation preserving diffeomorphism f of Σ and each $n \geq 2$ there is some m such that f has a fixed point in the moduli space of irreducible flat $SU(n^m)$ connections on Σ . Thirdly we prove that for all $n \geq 2$ there exists an integer m such that the m 'th power of any diffeomorphism fixes a certain point in the moduli space of irreducible flat $SU(n)$ connections on Σ .

1. INTRODUCTION

Let Σ be an oriented compact surface with or without boundary components. Consider the fundamental group of the surface, π_1 . Let $M(n)$ be the moduli space of irreducible flat $SU(n)$ -connections on Σ . Hence we have that

$$M(n) = \text{Hom}^{\text{irr}}(\pi_1, SU(n))/SU(n).$$

Let Γ be the mapping class group of Σ . Let f be an element of Γ . Now f acts on $M(n)$. We are interested in the following questions:

1. Which f 's have fixed points in $M(n)$?
2. Does Γ have a fixed point in $M(n)$?

We shall for certain n answer these questions affirmatively. One should remark that the fixed point set of any finite order or reducible finite order diffeomorphism can be described explicitly (see [1]). The first question is therefore only interesting for pseudo-Anosov diffeomorphisms. Answering yes to 2 for all n of course implies 1. It is however not quite clear if the techniques presented in this paper can establish that, though we can answer 1 positively for some nontrivial subgroups of Γ for all n .

We shall here analyse this problem using elementary finite group theory. We will show that for all primes p greater than 5 there is an integer m such that Γ has a fixed point in $M(p^m)$. We will also see that our technique can show that for all

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$n \geq 2$ there is some integer m (which may depend on n) and a point in $M(n)$ which is preserved by the m 'th power of any f in Γ .

In some particular cases, a positive answer to question 1 was obtained by Frohman in [2] (see also [3]). By using some involved cohomology calculations on the classifying space of certain gauge groups, Frohman gives an existence proof for fixed points for certain diffeomorphism of surfaces with 1 boundary component.

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2. INVARIANT NORMAL SUBGROUPS OF THE FUNDAMENTAL GROUP OF A SURFACE

Let Σ be a compact oriented surface with or without boundary components. Consider the fundamental group of the surface, $\pi_1 = \pi_1(\Sigma)$. Suppose f is an orientation preserving diffeomorphism of Σ . We are now interested in f -invariant normal finite index subgroups of π_1 . Suppose π is a normal subgroup of finite index of π_1 . Then $f(\pi)$ is again a normal subgroup of the same finite index as π . Let $\tilde{\pi}$ be the intersection

$$\tilde{\pi} = \bigcap_n f^n(\pi).$$

This intersection is actually a finite intersection, since there are only finitely many normal subgroups of a given index. In fact, the number of subgroups of index i of π_1 is bounded by the number of i -fold covers of Σ . Let m be the smallest integer such that

$$\tilde{\pi} = \bigcap_{n=0}^{m-1} f^n(\pi).$$

Hence we have a finite quotient group $G_f(\pi) = \pi_1/\tilde{\pi}$, on which there is an induced action of f . If we let

$$\tilde{\pi}_k = \bigcap_{n=0}^k f^n(\pi),$$

then we have the following diagram:

$$\begin{array}{ccccccc}
 & & 0 & & & & \\
 & & \downarrow & & & & \\
 & & f^{k+1}(\pi)/\tilde{\pi}_{k+1} & & & & \\
 & & \downarrow & & & & \\
 0 & \longrightarrow & \tilde{\pi}_k/\tilde{\pi}_{k+1} & \longrightarrow & \pi_1/\tilde{\pi}_{k+1} & \xrightarrow{q_1} & \pi_1/\tilde{\pi}_k \longrightarrow 0. \\
 & & & & \downarrow q_2 & & \\
 & & & & \pi_1/f^{k+1}(\pi) & & \\
 & & & & \downarrow & & \\
 & & & & 0 & &
 \end{array}$$

We have that (q_1, q_2) injects $\pi_1/\tilde{\pi}_{k+1}$ as a subgroup of $\pi_1/\tilde{\pi}_k \times \pi_1/f^{k+1}(\pi)$. Moreover, we also see that $\tilde{\pi}_k/\tilde{\pi}_{k+1}$ injects as a normal subgroup of $\pi_1/f^{k+1}(\pi)$, and f induces an isomorphism of $\pi_1/f^{k+1}(\pi)$ with π_1/π .

Now in order to be able to control the representation theory of $\pi_1/\tilde{\pi}$, we are going to assume that π_1/π is a simple group. We will see shortly that there are lots of examples where this is the case. The reason for this assumption is clear from the following lemma.

Lemma 2.1. *In the case where π_1/π is a simple group we have that*

$$\pi_1/\tilde{\pi} \cong \pi_1/\pi \times \pi_1/\pi \times \dots \times \pi_1/\pi \text{ (} m \text{ factors)}.$$

Proof. We shall prove the lemma by induction. Assume that

$$\pi_1/\tilde{\pi}_k \cong \pi_1/\pi \times \pi_1/\pi \times \dots \times \pi_1/\pi \text{ (} k+1 \text{ factors)}.$$

This assumption is clearly true when $k=0$. Since $\tilde{\pi}_k/\tilde{\pi}_{k+1}$ is a normal subgroup of $\pi_1/f^{k+1}(\pi)$, which is isomorphic to π_1/π , we have that $\tilde{\pi}_k/\tilde{\pi}_{k+1}$ must be either trivial or isomorphic to π_1/π . If $\tilde{\pi}_k/\tilde{\pi}_{k+1}$ is trivial, then $\tilde{\pi}_k = \tilde{\pi}$. So $k = m-1$, in which case we have proved the lemma. Assume then that $\tilde{\pi}_k/\tilde{\pi}_{k+1} \cong \pi_1/\pi$. From the vertical sequence in the above diagram, we then see that $\pi_1/\tilde{\pi}_{k+1}$ has order $|\pi_1/\pi|^{k+1} \times |\pi_1/\pi| = |\pi_1/\pi|^{k+2}$. But then (q_1, q_2) must be an isomorphism from $\pi_1/\tilde{\pi}_{k+1}$ to $\pi_1/\tilde{\pi}_k \times \pi_1/f^{k+1}(\pi)$. By induction we then get that

$$\pi_1/\tilde{\pi}_{k+1} \cong \pi_1/\pi \times \pi_1/\pi \times \dots \times \pi_1/\pi \text{ (} k+2 \text{ factors)}. \quad \square$$

Remark. We note that the m in the above lemma is clearly bounded by the number of principal π_1/π bundles over Σ , which in turn is bounded by the number of $|\pi_1/\pi|$ -sheeted covers of Σ .

Notice that in the above discussion, we could have defined

$$\overline{\pi} = \bigcap_{f \in \Gamma} f(\pi).$$

Exactly the same arguments show that there is some m such that

$$\pi_1/\overline{\pi} \cong \pi_1/\pi \times \pi_1/\pi \times \dots \times \pi_1/\pi \text{ (} m \text{ factors)}.$$

This way, we get a finite group $G_\Gamma(\pi) = \pi_1/\overline{\pi}$, on which there is an induced action of the mapping class group Γ .

Using the notation from the introduction, we see that in order to construct a fixed point for Γ 's action on $M(n)$, we just need to find an irreducible n -dimensional representation of the finite group $G_\Gamma(\pi)$, which is invariant under Γ . One way to construct such a situation without knowing anything about how Γ acts on $G_\Gamma(\pi)$, is to find a dimension in which $G_\Gamma(\pi)$ has only one irreducible representation. Suppose we can find a normal subgroup π of π_1 such that π_1/π is simple and has a unique irreducible representation in dimension p , where p is a prime, and such that π_1/π does not have any other irreducible representations of dimension p^n for any n . Then $G_\Gamma(\pi)$ will have a unique irreducible representation of dimension p^m . Hence we see that Γ will fix that corresponding point in $M(p^m)$.

It is now time to deliver on the promised normal subgroups.

Consider the alternating groups A_d , $d \geq 5$. These are all simple and can be given a presentation using two generators (see e.g. [5]). Since the free group on

two generators is the quotient of the fundamental group of any oriented surface of negative Euler-characteristic, we see there is a normal subgroup π^d of π_1 such that

$$A_d = \pi_1 / \pi^d.$$

Consider now the case where $d = p + 1$, where $p > 5$ is a prime. We then have the standard irreducible representation V of A_d of dimension p . Moreover, it is easy to prove that this is the only representation of dimension p (see e.g. p.67 in [4]).¹ By using the hook length formula one can very easily prove that the symmetric group S_d does not have any irreducible representations of dimension p^n or $2p^n$ for $n > 1$. If it did, we would have that

$$p^n = \frac{(p+1)!}{h_1} \text{ or } 2p^n = \frac{(p+1)!}{h_2},$$

where here $h_1, h_2 \in \mathbb{N}$ are the hook lengths of the representations. But from this and unique factorization we get that $n = 1$. Hence A_d cannot have any irreducible representations of dimension p^n for $n > 1$. (See Proposition 5.1 p.64 in [4] for the relation between the representations of A_d and S_d .)

Hence (π^d, A_d, V) satisfies all the requirements to guarantee that there is an $m = m(d)$ which is bounded by the number of $\frac{d!}{2}$ -sheeted covers of Σ , such that Γ has a fixed point in $M(p^m)$.

Hence we have proved

Theorem 2.1. *For any prime $p > 5$ there is an $m = m(p)$ such that Γ has a fixed point in $M(p^m)$.*

Now, in the case where we are just interested in one element $f \in \Gamma$, we can actually say a bit more. From the proof of lemma 2.1 we get in fact an f equivariant identification

$$G_f(\pi) \cong \pi_1 / \pi \times \pi_1 / f(\pi) \times \dots \times \pi_1 / f^{m-1}(\pi),$$

which means that when we use the identification

$$G_f(\pi) \cong \pi_1 / \pi \times \pi_1 / \pi \times \dots \times \pi_1 / \pi \text{ (} m \text{ factors),}$$

f just becomes

$$f(g_1, \dots, g_m) = (f^m(g_m), g_1, g_2, \dots, g_{m-1}).$$

Hence, if we have an irreducible n -dimensional representation ρ of π_1 / π whose isomorphism class is preserved by f^m , then the isomorphism class of $\rho_{G_f} = \rho \otimes \dots \otimes \rho$ is preserved by f . If ρ is the only irreducible representation of π_1 / π of dimension n , the isomorphism class of ρ must be preserved by f^m .

To construct some concrete examples, consider the normal subgroup π^d of π_1 considered above. Now d is any integer greater than or equal to 5, but not equal to 6. As we mentioned before, there is only one irreducible representation of dimension $n = d - 1$ of A_d . Let ρ_n be that representation. $\rho_{G_n} = \rho_n \otimes \dots \otimes \rho_n$ then gives a fixed point for f in $M(n^m)$.

We can alternatively fix the normal subgroup π of π_1 and then change f to a power of f . We no longer have to worry about what f maps π to, since we will simply replace f by a power of f which fixes π . The requirement that π_1 / π should

¹This is not the case for A_6 , so we have excluded this case.

be simple is no longer needed. We can for example make the following construction. Let π be a normal subgroup of π_1 such that

$$\pi_1/\pi = S_3.$$

We then know that for some m less than the number of 6-sheeted covers of Σ , we have

$$f^m(\pi) = (\pi).$$

Now S_3 has a unique unitary irreducible 2-dimensional representation. Then this representation gives us a point in $M(2)$ which is fixed by f^m for all $f \in \Gamma$. (We have here used the same symbol $M(2)$ to represent the $U(2)$ and the $SU(2)$ moduli space.) The smallest m that does this will of course depend on f . However, if we let l be the number of normal subgroups of π_1 whose quotient is S_3 , then f^l preserves the given point in $M(2)$.

Now notice that each f in Γ permutes the l normal subgroups with S_3 quotient; hence we get a homomorphism from Γ to S_l . The kernel of this homomorphism is of course a normal subgroup of finite index of Γ . For the elements f in this normal subgroup, we can by definition take $m = 1$ above, i.e. they fix the above described point in $M(2)$.

To get such fixed points in the moduli spaces $M(n)$ where n is not 2, we just repeat the above construction, where we replace S_3 with some group which has a unique irreducible representation in dimension n . For $n \neq 5$ just take the unique irreducible representation of A_{n+1} of dimension n . For $n = 5$ take the unique irreducible 5-dimensional representation of A_4 .

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