CHARACTERIZATIONS OF PSEUDODIFFERENTIAL OPERATORS ON THE CIRCLE

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(Communicated by Palle E. T. Jorgensen)

ABSTRACT. Globally defined operators are shown to be equivalent to the classical pseudodifferential operators on the circle. A characterization of the smooth operators for the regular representation of $\mathbb{S}^1$ is also given.

1. INTRODUCTION

The introduction of a symbol into the Fourier inversion formula on the circle defines a “discrete” pseudodifferential operator, as in the case of the pseudodifferential operators on $\mathbb{R}$. Several classes of these globally defined operators have been shown to be equivalent to classical pseudodifferential operators on $\mathbb{S}^1$, the first (unpublished) result of this kind being attributed by M. S. Agranovich [2] to L. R. Volevich. Theorems on this theme are found, for example, in the papers [1], [2], [11] and [8], where also many applications are developed or mentioned.

The smooth vectors for the canonical action of the Heisenberg group on $\mathcal{L}(L^2(\mathbb{R}))$, the algebra of bounded operators on $L^2(\mathbb{R})$, were characterized by H. O. Cordes [5] as those pseudodifferential operators whose symbols have bounded derivatives of all orders. Similar criteria hold for other Lie groups [4, 5]. These results have been applied to nonlinear problems [9] and are also related to deformations of C^* -algebras [10].

In Section 2 of this paper, the discrete pseudodifferential operators are studied in detail. Theorem 2 is then a consequence of H. O. Cordes’ smoothness theorem and of the equivalence between certain local pseudodifferential operators and a class of discrete pseudodifferential operators. The smooth vectors for the canonical action of the translation group on $\mathcal{L}(L^2(\mathbb{S}^1))$ are thus characterized as those discrete pseudodifferential operators whose symbols have all derivatives bounded in $\mathbb{S}^1$, uniformly with respect to the discrete variable. This means, in particular, that the class $Op\psi_{dt_0}$ defined in Section 3 is a Ψ^* -algebra in the sense of B. Gramsch [6]. E. Schrohe’s results [12] imply, in particular, that the class $Op\psi_{dc_0}$ defined in Section 2 is also a Ψ^* -algebra.

The present results can be extended to the n -dimensional torus with a few minor changes of definitions and notation. For the sake of transparency of the argument, however, only the case of $n = 1$ is exposed. It must be pointed out that our

Received by the editors November 14, 1995.

1991 *Mathematics Subject Classification*. Primary 47G30; Secondary 35S05, 58G15.

Theorem 1 is not completely original (see W. McLean's Theorem 4.4 of [8]), but its proof is somewhat simpler.

2. DISCRETE PSEUDODIFFERENTIAL OPERATORS

The symbols considered here are of the form $b(x, j)$, $x \in \mathbb{R}$ and $j \in \mathbb{Z}$, with $b(\cdot, j)$ being a 2π -periodic smooth function for each j . Denoting by ∇^l the l -th power of the difference operator acting on sequences, $(\nabla a)_j = a_{j+1} - a_j$, and by $\langle \cdot \rangle$ the function $\langle t \rangle = \sqrt{1+t^2}$, $t \in \mathbb{R}$, we say that b is a *discrete classical symbol of order m* if, for every pair of non-negative integers (k, l) , there is a constant C_{kl} such that

$$(1) \quad |\nabla^l \partial_x^k b(x, j)| \leq C_{kl} \langle j \rangle^{m-l}, \quad j \in \mathbb{Z}.$$

The smallest C_{kl} for which (1) holds is denoted by $\|b\|_{kl}$ and the class of all b satisfying (1) is denoted by ψ_{dc_m} .

A *discrete pseudodifferential operator* B acts on functions on the circle $\mathbb{S}^1$ and is defined by the formula

$$(2) \quad Bv(e^{ix}) = \sum_{j \in \mathbb{Z}} e^{ijx} b(x, j) v_j,$$

where v_j denotes the Fourier coefficient $v_j = \frac{1}{2\pi} \int_{-\pi}^{\pi} e^{-ijx} v(e^{ix}) dx$.

Proposition 1. *If b is a discrete classical symbol of order $m \in \mathbb{Z}$, then the equation (2) defines a continuous operator $B : C^\infty(\mathbb{S}^1) \rightarrow C^\infty(\mathbb{S}^1)$. If $m \leq 0$, then B extends to a bounded operator on $L^2(\mathbb{S}^1)$ whose norm satisfies*

$$(3) \quad \|B\| \leq \max\{\|b\|_{2,0}, \|b\|_{0,0}\} \cdot \sum_{j \in \mathbb{Z}} \langle j \rangle^{-2}.$$

Proof. That B is a well-defined continuous operator on $C^\infty(\mathbb{S}^1)$ follows from

$$\sum_{j \in \mathbb{Z}} |j^l \partial_x^k b(x, j) v_j| \leq \|b\|_{k,0} \cdot \sup |(1 - \partial_x^2)^p v(e^{ix})| \cdot \sum_{j \in \mathbb{Z}} \langle j \rangle^{m+l-2p},$$

where, for each l , p is chosen so that $2p - m - l \geq 2$.

In order to prove the L^2 -boundedness, one needs to estimate the matrix elements $b_{kj} = (2\pi)^{-1} \int_{-\pi}^{\pi} b(x, j) e^{i(j-k)x} dx$. Using that $e^{i(j-k)x} = \langle j-k \rangle^{-2} (1 - \partial_x^2) e^{i(j-k)x}$ and integrating by parts, one gets $|b_{kj}| \leq \max\{\|b\|_{2,0}, \|b\|_{0,0}\} \langle j-k \rangle^{-2}$. It follows that the sum $\sum_{j \in \mathbb{Z}} |b_{kj}|$ is bounded by the constant on the right-hand side of (3) for every k and that the corresponding statement with the roles of k and j interchanged is also true. The proposition follows from Schur's lemma (see, for example, [3]). $\square$

The class of all operators B with discrete classical symbols of order m is denoted by $Op\psi_{dc_m}$. Next result states that it coincides with the usual class of pseudodifferential operators on the circle.

Theorem 1. *An operator $B : C^\infty(\mathbb{S}^1) \rightarrow C^\infty(\mathbb{S}^1)$ belongs to $Op\psi_{dc_m}$, $m \in \mathbb{Z}$, if and only if it is a pseudodifferential operator of order m on $\mathbb{S}^1$ in the usual sense; i.e., $B \in \Psi^m(\mathbb{S}^1)$ in the notation of [7], or $B \in LC_m(\mathbb{S}^1)$ in the notation of [4].*

Proof. Let $B \in Op\psi_{dc_m}$ be given. For each $\theta \in \mathbb{R}$, let us consider the chart $\chi : \mathbb{S}^1 \setminus \{-e^{i\theta}\} \rightarrow (-\pi + \theta, \pi + \theta)$, $\chi(e^{ix}) = x$. Given φ and ω in $C^\infty(\mathbb{S}^1)$, both vanishing on a neighborhood of $-e^{i\theta}$, let B_θ denote the operator on the Schwartz space $\mathcal{S}(\mathbb{R})$ defined by $(B_\theta u) \circ \chi = \omega B[(\varphi u) \circ \chi]$, where $\lambda_\theta = \lambda \circ \chi^{-1}$. In order

to prove that $B \in \Psi^m(\mathbb{S}^1)$, it is enough to show that B_θ is a pseudodifferential operator on $\mathbb{R}$ whose symbol satisfies the estimates on (7).

The distribution kernel of B_θ as an operator on the Schwartz space $\mathcal{S}(\mathbb{R})$ equals

$$\frac{1}{2\pi} \sum_{j \in \mathbb{Z}} e^{i(x-y)j} b(x, j) \varphi_\theta(x) \omega_\theta(y),$$

where the sum converges in distribution sense. If B_θ is a pseudodifferential operator on $\mathbb{R}$, its symbol must then be given by (see [4], Chapter 1, for example)

$$(4) \quad a(x, \xi) = \frac{\varphi_\theta(x)}{2\pi} \sum_j b(x, j) \int e^{i(j-\xi)z} \omega_\theta(x-z) dz;$$

and conversely, if we show that this expression defines a proper symbol, then the equality $B_\theta = a(x, D)$ will be established.

The domain of the integral in (4) may be taken as the interval $[-2\pi+\delta, 2\pi-\delta]$, for some $\delta > 0$, because the supports of φ_θ and of ω_θ are both compact and contained in $(-\pi+\theta, \pi+\theta)$. The sum in equation (4) is absolutely convergent, since the integral appearing on the right-hand side is of the order of $\langle j-\xi \rangle^{-2p}$, for any integer p . The derivatives with respect to ξ may be brought inside the sum, since the absolute convergence also holds if we replace $\omega_\theta(x-z)$ by $z^l \omega_\theta(x-z)$, $l \geq 1$. This power of z and the use of the formula $e^{ijz} = (e^{iz} - 1)^{-l} \nabla^l e^{ijz}$ allow us to do partial summation as follows:

$$\begin{aligned} \partial_\xi^l a(x, \xi) &= \frac{\varphi_\theta(x)}{2\pi} \sum_j b(x, j) \nabla^l \left(\int_{-2\pi+\delta}^{2\pi-\delta} e^{i(j-\xi)z} \frac{(-iz)^l}{(e^{iz} - 1)^l} \omega_\theta(x-z) dz \right) = \\ &= (-1)^l \frac{\varphi_\theta(x)}{2\pi} \sum_j [\nabla^l b(x, j-l)] \int_{-2\pi-\delta}^{2\pi+\delta} e^{i(j-\xi)z} \frac{(-iz)^l}{(e^{iz} - 1)^l} \omega_\theta(x-z) dz. \end{aligned}$$

Using that $e^{i(j-\xi)z} = \langle j-\xi \rangle^{-2p} (1 - \partial_z^2)^p e^{i(j-\xi)z}$, integrating by parts and using also (1), it follows that

$$(5) \quad |\partial_\xi^l a(x, \xi)| \leq C \sum_j \langle j \rangle^{m-l} \langle j-\xi \rangle^{-2p},$$

for any non-negative integer p , with C equal to

$$2 \|b\|_{0,l} \sup_j \frac{\langle j-l \rangle^{m-l}}{\langle j \rangle^{m-l}} \sup_{|x-\theta| < \pi, |z| < 2\pi-\delta} \left| \varphi_\theta(x) (1 - \partial_z^2)^p \frac{z^l \omega_\theta(x-z)}{(e^{iz} - 1)^l} \right|.$$

Using now that $\langle x \rangle \leq 2 \langle y \rangle \langle x-y \rangle$, for all real x and y , it follows that

$$(6) \quad |\partial_\xi^l a(x, \xi)| \leq C 2^{l(m-l)} \langle \xi \rangle^{m-l} \sum_{j \in \mathbb{Z}} \langle j-\xi \rangle^{l(m-l)-2p}.$$

If p is large enough, this last sum defines a continuous and periodic function of ξ and is then bounded by a constant.

The previous argument also works after (4) is differentiated with respect to x . It then follows that a is a classical symbol of order m , i.e., it satisfies the estimate

$$(7) \quad |\partial_x^k \partial_\xi^l a(x, \xi)| \leq C_{k,l} \langle \xi \rangle^{m-l}.$$

Conversely, let $B \in \Psi^m(\mathbb{S}^1)$ be given. By means of a suitable choice of a partition of unity, it can be seen that it is enough to show that $B^\dagger = \varphi(M) B \omega(M)$ is in

$Op\psi_{dc_m}$ for any pair (φ, ω) of functions in $C^\infty(\mathbb{S}^1)$ both vanishing on a neighborhood of some point $e^{-i\theta}$. We have denoted by $\zeta(M)$ the operator multiplication by ζ . Since $B^\dagger$ is continuous on $C^\infty(\mathbb{S}^1)$, if it belongs to $Op\psi_{dc_m}$ its discrete symbol must then be given by

$$(8) \quad b(x, j) = e^{-ijx} B^\dagger e_j(e^{ix}),$$

where $e_j(z) = z^j$, $z \in \mathbb{S}^1$; and conversely, if equation (8) defines a discrete symbol in ψ_{dc_m} , the corresponding operator in $Op\psi_{dc_m}$ must then be equal to $B^\dagger$. All that is left is to show that b defined on (8) indeed satisfies (1).

By hypothesis, $B^\dagger$ is locally given as a pseudodifferential operator on $\mathbb{R}$ with symbol a satisfying (7). It then follows that

$$(9) \quad b(x, j) = \frac{1}{2\pi} \int_{-\infty}^{+\infty} a(x, \xi) \left(\int_{-\pi+\theta}^{\pi+\theta} e^{i(x-y)(\xi-j)} \rho(y) dy \right) d\xi$$

where $\rho \in C_0^\infty(-\pi + \theta, \pi + \theta)$ is equal to one on the support of ω_θ . Using that $\nabla^l e^{ijz} = e^{ijz} (e^{iz} - 1)^l$ and that $e^{i(x-y)\xi} = i^{-l} (x-y)^{-l} \partial_\xi^l e^{i(x-y)\xi}$, we get:

$$\nabla^l b(x, j) = \frac{1}{2\pi} \int_{-\infty}^{+\infty} a(x, \xi) \partial_\xi^l \left(\int_{-\pi+\theta}^{\pi+\theta} e^{i(x-y)(\xi-j)} \left[\frac{e^{i(y-x)} - 1}{i(x-y)} \right]^l \rho(y) dy \right) d\xi.$$

In order to integrate by parts in ξ , we need the estimate

$$\left| \partial_\xi^q \int_{-\pi+\theta}^{\pi+\theta} e^{i(x-y)(\xi-j)} \left[\frac{e^{i(y-x)} - 1}{i(x-y)} \right]^l \rho(y) dy \right| \leq 2\pi A_{p,l,q}(x) \langle \xi - j \rangle^{-2p},$$

which holds for any integers $p \geq 0$ and $0 \leq q \leq l$, with

$$A_{p,l,q}(x) = \sup_{|y-\theta| < \pi} \left| (1 - \partial_y^2)^p [(e^{i(y-x)} - 1)^l (x-y)^{q-l} \rho(y)] \right|.$$

If p is chosen so that $m - l - 2p \leq -2$, then it follows from (7) that the desired integration by parts is legitimate and that

$$(10) \quad |\nabla^l b(x, j)| \leq C_{0,l} A_{p,l,0}(x) \int_{-\infty}^{+\infty} \langle \xi \rangle^{m-l} \langle \xi - j \rangle^{-2p} d\xi.$$

Since $A_{p,l,0}$ depends continuously on x and $b(x, j)$ is periodic in x , the inequality on (10) also holds with a constant independent of x . Using once more the inequality $\langle \xi \rangle^{m-l} \leq 2^{|m-l|} \langle j \rangle^{m-l} \langle \xi - j \rangle^{|m-l|}$ and requiring that $-2p + |m-l| \leq -2$, it follows that the integral on the right-hand side of (10) is bounded by the constant $2^{|m-l|} \langle j \rangle^{m-l} \int \langle \xi \rangle^{-2} d\xi$. This proves (1) for $k = 0$.

When (9) is differentiated with respect to x , some of the derivatives will fall on $a(x, \xi)$ and some on $e^{i(x-y)(\xi-j)}$. Those on the symbol cause no trouble, since any derivative of a with respect to x also satisfies (7) with different constants. Those on the exponential may be transformed into derivatives with respect to y , since $\partial_x e^{i(x-y)(\xi-j)} = -\partial_y e^{i(x-y)(\xi-j)}$. An integration by parts then brings the derivatives to ρ and thus the same argument used for the case when $k = 0$ will finish the proof. $\square$

3. SMOOTHNESS CRITERION

Let us denote by ψ_{dt_m} , $m \in \mathbb{Z}$, the class of the symbols $b(x, j)$ satisfying (1) possibly only for $l = 0$. Proposition 1 still holds for $b \in \psi_{dt_m}$, since only the finiteness of the seminorms $\|\cdot\|_{k,0}$, $k \geq 0$, is needed in the proof. Denoting by $Op\psi_{dt_m}$ the class of all corresponding B defined by (2), it is also clear that a weaker version of Theorem 1 is true; namely, that a continuous $B : C^\infty(\mathbb{S}^1) \rightarrow C^\infty(\mathbb{S}^1)$ belongs to $Op\psi_{dt_m}$ if and only if, for every real θ and every pair of smooth functions (φ, ω) vanishing on a neighborhood of $e^{-i\theta}$, the operator B_θ defined at the beginning of the proof of Theorem 1 is a pseudodifferential operator on $\mathbb{R}$ with symbol a satisfying

$$(11) \quad |\partial_x^k \partial_\xi^l a(x, \xi)| \leq C_{k,l} \langle \xi \rangle^m.$$

Since the support of a projects onto a compact subset of the x -axis, such a symbol belongs to $\psi_{t(m, -\infty)}$, as defined in [4].

Let T_z , $z \in \mathbb{R}$, denote the unitary operator on $L^2(\mathbb{S}^1)$ defined by $(T_z u)(e^{ix}) = u(e^{i(x-z)})$. The following is now a consequence of Theorem VIII.2.1 of [4].

Theorem 2. *Given B a bounded operator on $L^2(\mathbb{S}^1)$, the map*

$$(12) \quad \begin{array}{ccc} \mathbb{R} & \longrightarrow & \mathcal{L}(L^2(\mathbb{S}^1)), \\ z & \longmapsto & T_z^{-1} B T_z \end{array}$$

has operator-norm derivatives of all orders if and only if $B \in Op\psi_{dt_0}$.

Proof. If $B \in \mathcal{L}(L^2(\mathbb{S}^1))$ is such that the map on (12) is smooth, the same is true for $\varphi(M)B\omega(M)$, where φ and ω are smooth functions on $\mathbb{S}^1$. If both functions vanish on a neighborhood of some point, the operators $B(z) = T_z^{-1}\varphi(M)B\omega(M)T_z$ may be viewed as operators on $L^2(\mathbb{R})$, defining an operator-valued function which is norm-smooth with respect to z , for small $z \in \mathbb{R}$. We must show that conjugation of $B(z)$ by the multiplication operators e^{iyM} is also $\mathcal{L}(L^2(\mathbb{R}))$ -smooth with respect to y . Let φ_1 and ω_1 be smooth functions of supports slightly larger than those of φ and ω and identically equal to one on the supports of φ and of ω , respectively. We then have

$$e^{iyM} B(0) e^{-iyM} = [e^{iyM} \varphi_1(M)] B(0) [e^{-iyM} \omega_1(M)],$$

which is clearly smooth with respect to y . Using the commutation relation between T_z and e^{iyM} , it follows that

$$\begin{array}{ccc} \mathbb{R}^2 & \longrightarrow & \mathcal{L}(L^2(\mathbb{R})), \\ (z, y) & \longmapsto & e^{-iyM} T_{-z} (\varphi(M) B \omega(M)) T_z e^{iyM} \end{array}$$

is norm smooth. Cordes' theorem quoted above then implies that $\varphi(M)B\omega(M)$ is locally given as a pseudodifferential operator on $\mathbb{R}$ with symbol $a \in \psi_{t(0,0)}$. It follows that B is continuous on $C^\infty(\mathbb{S}^1)$ and thus that B is in $Op\psi_{dt_0}$.

Conversely, a given $B \in Op\psi_{dt_0}$ can be written as a sum of operators of the type $\varphi(M)B\omega(M)$, with φ and ω vanishing on a neighborhood of some point of the circle. Each of these operators is locally given as a pseudodifferential operator with symbol in $\psi_{t(0,0)}$. Again by Cordes' smoothness theorem, these local operators are smooth under the Heisenberg group action, in particular under the translation group action on $\mathcal{L}(L^2(\mathbb{R}))$. This shows that each of the operators $\varphi(M)B\omega(M)$ is $\mathcal{L}(L^2(\mathbb{S}^1))$ -smooth under the action of the translation group of $\mathbb{S}^1$. $\square$

ACKNOWLEDGEMENT

The author thanks Prof. H. O. Cordes for suggesting the subject and for many helpful discussions.

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