BEALS-CORDES-TYPE CHARACTERIZATIONS OF PSEUDODIFFERENTIAL OPERATORS

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ABSTRACT. We show that, if U is the representation of $SO_e(n+1, 1)$ on $L^2(S^n)$ given by (2.11), and P is a bounded operator on $L^2(S^n)$, then P belongs to $OPS_{1,0}^0(S^n)$ if and only if

$$P(g) = U(g)PU(g)^{-1}$$

is a C^∞ function on $SO_e(n+1, 1)$ with values in the Banach space $\mathcal{L}(L^2(S^n))$.

INTRODUCTION

Let M be a compact C^∞ manifold. Denote by $OPS_{1,0}^m(M)$ the space of pseudodifferential operators on M , whose symbols in local coordinates satisfy

$$|D_x^\beta D_\xi^\alpha p(x, \xi)| \leq C_{\alpha\beta} \langle \xi \rangle^{m-|\alpha|},$$

and denote by $OPS^m(M)$ the subspace of classical pseudodifferential operators, whose symbols have the asymptotic behavior

$$p(x, \xi) \sim p_m(x, \xi) + p_{m-1}(x, \xi) + \cdots, \quad |\xi| \rightarrow \infty,$$

with $p_{m-j}(x, \xi)$ homogeneous of degree $m-j$ in ξ .

It was demonstrated in [B], [C], and [D] that, if $P: L^2(M) \rightarrow L^2(M)$, then P belongs to $OPS_{1,0}^0(M)$ if and only if, for arbitrary $A_j \in OPS^1(M)$, and any $N \in \mathbb{Z}^+$,

$$\text{ad } A_N \cdots \text{ad } A_1 \cdot P: L^2(M) \rightarrow L^2(M),$$

where $\text{ad } A_j \cdot B = [A_j, B]$.

It is desirable to have some alternative characterizations, and we give one in §2 of this paper, which has as a special case the following (when $M = S^n$, the n -dimensional sphere):

Theorem. *If U is the representation of $SO_e(n+1, 1)$ on $L^2(S^n)$ given by (2.11), and P is a bounded operator on $L^2(S^n)$, then P belongs to $OPS_{1,0}^0(S^n)$ if and only if*

$$P(g) = U(g)PU(g)^{-1}$$

is a C^∞ function on $SO_e(n+1, 1)$ with values in the Banach space $\mathcal{L}(L^2(S^n))$.

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It is technically useful that this characterization involves smoothness on a finite dimensional Lie group. One implication is that the group of invertible operators in $OPS_{1,0}^0(M)$ forms a smooth tame Frechét Lie group (in the terminology of [H]), by arguments such as used in [P]. This in turn leads to a simplification and elucidation of some of the pseudodifferential operator estimates in [GY]. We plan to pursue this matter further, in another paper.

1. GENERALITIES; $\mathfrak{A}$ -SMOOTH OPERATORS

Let $\mathfrak{A} = \{A_j\}$ be a collection of operators in $OPS^1(M)$, for some compact smooth Riemannian manifold M . We will make the following two hypotheses on $\mathfrak{A}$:

$$(1.1) \quad A_j^* = A_j, \quad \text{mod } OPS^0(M);$$

$$(1.2) \quad \text{some finite sum } L = \sum A_j^2 \text{ is elliptic in } OPS^2(M).$$

An operator $P \in \mathcal{L}(L^2(M))$ will be called $\mathfrak{A}$ -smooth provided

$$(1.3) \quad (\text{ad } A)^J P \text{ is bounded on } L^2(M),$$

for each $(\text{ad } A)^J = \text{ad } A_{j_l} \cdots \text{ad } A_{j_1}$, where $\text{ad } A \cdot P = [A, P]$. We use a common multi-index notation: $J = (j_l, \dots, j_1)$, we also set $|J| = l$. A priori, the hypothesis $P: L^2(M) \rightarrow L^2(M)$ implies that $(\text{ad } A)^J P: C^\infty(M) \rightarrow \mathcal{D}'(M)$.

We mention a few very simple results, with almost trivial proofs.

Lemma 1.1. *The set $\mathcal{S}_{\mathfrak{A}}$ of $\mathfrak{A}$ -smooth operators is an algebra, containing $OPS_{1,0}^0(M)$.*

Proof. That $\mathcal{S}_{\mathfrak{A}}$ is an algebra follows from the identity

$$(1.4) \quad (\text{ad } A_j)(P_1 P_2) = P_1(\text{ad } A_j \cdot P_2) + (\text{ad } A_j \cdot P_1)P_2.$$

□

Lemma 1.2. *Let $\mathfrak{A}' = \{A_j + B_j\}$, $B_j \in OPS^0(M)$. If P is $\mathfrak{A}$ -smooth, it is also $\mathfrak{A}'$ -smooth.*

Lemma 1.3. *If P is $\mathfrak{A}$ -smooth, so is P^* .*

Proof. We have $\text{ad } A_j \cdot P^* = -(\text{ad } A_j^* \cdot P)^*$, hence

$$(1.5) \quad (\text{ad } A)^J P^* = (-1)^{|J|} [(\text{ad } A^*)^J P]^*.$$

□

Lemma 1.4. *If P is $\mathfrak{A}$ -smooth, so is each $(\text{ad } A)^J P$.*

Lemma 1.5. *If P is $\mathfrak{A}$ -smooth, then P and $(\text{ad } A)^J P$ preserve each Sobolev space $H^s(M)$.*

Proof. We have

$$A_k P = P A_k + \text{ad } A_k \cdot P.$$

Since the left side maps $L^2(M)$ to $H^{-1}(M)$, we have

$$P A_k: L^2(M) \rightarrow H^{-1}(M).$$

By (1.2) it follows that $P: H^{-1}(M) \rightarrow H^{-1}(M)$. That $P: H^1(M) \rightarrow H^1(M)$ follows by duality, given Lemma 1.3. The rest follows by iteration and interpolation.

□

We next note the approximate commutativity of P with L . Indeed,

$$(1.6) \quad [L, P] = \sum \{(\operatorname{ad} A_j \cdot P)A_j + A_j(\operatorname{ad} A_j \cdot P)\}.$$

If P is $\mathfrak{A}$ -smooth, this maps $H^s(M)$ to $H^{s-1}(M)$. Hence

$$(1.7) \quad \begin{aligned} LPL^{-1} - P &= [L, P]L^{-1}, \\ L^{-1}PL - P &= L^{-1}[P, L], \end{aligned}$$

with each right side mapping $H^s(M)$ into $H^{s+1}(M)$.

If P is $\mathfrak{A}$ -smooth, it can often be established that P is $\tilde{\mathfrak{A}}$ -smooth, for a much bigger collection of operators $\tilde{\mathfrak{A}}$. We make some simple comments on this phenomenon here; more incisive results will be given in §2. The next result follows immediately from Jacobi's identity

$$(1.8) \quad \operatorname{ad} A_1 \operatorname{ad} A_2 - \operatorname{ad} A_2 \operatorname{ad} A_1 = \operatorname{ad}[A_1, A_2].$$

Lemma 1.6. *Let $\mathfrak{L}\mathfrak{A}$ be the Lie algebra generated by $\mathfrak{A}$. If P is $\mathfrak{A}$ -smooth, then it is $\mathfrak{L}\mathfrak{A}$ -smooth.*

Generalizing the class $\mathcal{S}_{\mathfrak{A}}$ of $\mathfrak{A}$ -smooth operators let us say, for an operator P acting on $\mathcal{D}'(M)$ and preserving $C^\infty(M)$, that

$$(1.9) \quad P \in \mathcal{S}_{\mathfrak{A}}^m \Leftrightarrow (\operatorname{ad} A)^J P: H^s(M) \rightarrow H^{s-m}(M),$$

for all commutators as in (1.3). Thus $\mathcal{S}_{\mathfrak{A}}^0$ is what is denoted $\mathcal{S}_{\mathfrak{A}}$ above. Parallel to Lemma 1.1, we clearly have

$$(1.10) \quad P_j \in \mathcal{S}_{\mathfrak{A}}^{m_j} \Rightarrow P_1 P_2 \in \mathcal{S}_{\mathfrak{A}}^{m_1+m_2},$$

and

$$(1.11) \quad OPS_{1,0}^m(M) \subset \mathcal{S}_{\mathfrak{A}}^m.$$

In particular, for an operator $E: C^\infty(M) \rightarrow \mathcal{D}'(M)$,

$$(1.12) \quad EL \in \mathcal{S}_{\mathfrak{A}}^m \Rightarrow E \in \mathcal{S}_{\mathfrak{A}}^{m-2},$$

since $L^{-1} \in OPS^{-2}(M)$. Consequently we see that, if $\mathfrak{A}_b$ is the set of A_j over which we sum in (1.2),

$$(1.13) \quad EA_j \in \mathcal{S}_{\mathfrak{A}}^m \text{ for all } A_j \in \mathfrak{A}_b \Rightarrow E \in \mathcal{S}_{\mathfrak{A}}^{m-1}.$$

Now suppose $B_l \in OPS^0(M)$ are such that

$$(1.14) \quad B_l A_j \in \mathfrak{A} \text{ for all } A_j \in \mathfrak{A}_b.$$

Since

$$(1.15) \quad (\operatorname{ad} B_l \cdot P)A_j = \operatorname{ad}(B_l A_j) \cdot P - B_l(\operatorname{ad} A_j \cdot P),$$

we deduce that $\operatorname{ad} B_l \cdot P \in \mathcal{S}_{\mathfrak{A}}^{m-1}$ if $P \in \mathcal{S}_{\mathfrak{A}}^m$. We hence deduce the following:

Proposition 1.7. *If $(\operatorname{ad} C)^J = \operatorname{ad} C_{j_1} \cdots \operatorname{ad} C_{j_1}$, where μ of the C_j 's are B_j 's, satisfying (1.14), and the rest are A_j 's, in $\mathfrak{A}$, then*

$$(1.16) \quad P \in \mathcal{S}_{\mathfrak{A}}^m \Rightarrow (\operatorname{ad} C)^J P \in \mathcal{S}_{\mathfrak{A}}^{m-\mu}.$$

We note that this result applies when $\mathfrak{A}$ consists of all smooth vector fields on M and the B_l are multiplication operators, by smooth functions on M . In this case, Proposition 1.7 is one simple step in the well known result of Cordes (et al.) that $\mathcal{S}_{\mathfrak{A}}$ is precisely $OPS_{1,0}^0(M)$.

2. U -SMOOTH OPERATORS; A CHARACTERIZATION OF $OPS_{1,0}^0(M)$

Let M be a compact smooth manifold, and G a Lie group, with a strongly continuous unitary representation $U(g)$ on $L^2(M)$. Let $\{X_j: 1 \leq j \leq m\}$ be a basis of the Lie algebra $\mathfrak{g}$. Suppose $dU(X_j) = iA_j \in OPS^1(M)$. Also suppose that, if Δ is the Laplace operator on G , with a left invariant metric, then $dU(\Delta) = L$ is an elliptic operator in $OPS^2(M)$; one could take $L = -\sum A_j^2$.

Let $P \in \mathcal{L}(L^2(M))$. We say P is U -smooth if

$$(2.1) \quad P(t) = U(t)PU(t)^{-1}$$

is a C^∞ function of $t \in G$, with values in $\mathcal{L}(L^2(M))$. If P is U -smooth, it follows that P is $\mathfrak{A}$ -smooth, with $\mathfrak{A} = \{A_1, \dots, A_m\}$, i.e.,

$$\text{ad } A_{j_l} \cdots \text{ad } A_{j_1} \cdot P: L^2(M) \rightarrow L^2(M),$$

for all $j_1, \dots, j_l$. We aim to show that, under condition (2.4) given below, we have boundedness of commutators when the A_ν are replaced by arbitrary operators in $OPS^1(M)$, so that the criterion discovered by [B], [C], and [D] will apply.

Identify a small neighborhood $\mathcal{O}$ of the identity element $e \in G$ with a neighborhood of 0 in $\mathfrak{g}$, via the exponential map. Suppose $\varphi \in \mathcal{E}'(\mathcal{O})$ is a classical conormal distribution, with singularity at the origin, of the form $\varphi = \hat{f}$, with f in the symbol space $S^\mu(\mathfrak{g}^*)$. Set

$$(2.2) \quad f(A) = U(\hat{f}) = \int \hat{f}(t)\kappa(t)U(t)dt,$$

where $\kappa(t)dt$ is Haar measure (in exponential coordinates). For $\mu \ll 0$, this is readily verified to be smoothing, via the following simple result.

Lemma 2.1. *If $\varphi \in C_0^{2k}(G)$, then $U(\varphi): H^s(M) \rightarrow H^{s+2k}(M)$, for all $s \in [-2k, 0]$.*

Proof. For $s = 0$, this follows from the identity

$$L^k U(\varphi) = U(\Delta^k \varphi).$$

Then we have the result for $s = -2k$ by duality, and the rest by interpolation. $\square$

A more precise analysis of (2.2) is provided by the following result, proved in Appendix B (pp. 163–165) of [T3], following material developed in [T1] and in Chapter 12 of [T2].

Lemma 2.2. *If $f \in S^\mu(\mathfrak{g}^*)$, with $\hat{f}$ supported in $\mathcal{O}$, then $f(A) \in OPS^\mu(M)$, and the principal symbol of $f(A)$ is equal to $f(a_1, \dots, a_m)$ on $T^*M \setminus 0$, where $a_j(x, \xi)$ is the principal symbol of A_j .*

We now state our hypotheses on the action U . Consider the ‘moment map’

$$(2.3) \quad \Phi(x, \xi) = (a_1(x, \xi), \dots, a_m(x, \xi)); \quad \Phi: T^*M \setminus 0 \rightarrow \mathbb{R}^m \setminus 0.$$

The hypothesis is:

$$(2.4) \quad \Phi: T^*M \setminus 0 \rightarrow \mathbb{R}^m \setminus 0 \text{ is an imbedding.}$$

An equivalent statement is that for any $q_\mu \in S^\mu(T^*M \setminus 0)$, homogeneous of degree μ , there exists a homogeneous $f_\mu \in S^\mu(\mathfrak{g}^*)$ such that

$$q_\mu(x, \xi) = f_\mu(a_1(x, \xi), \dots, a_m(x, \xi)).$$

In light of Lemma 2.2, we see that, if $Q \in OPS^\mu(M)$ has principal symbol q_μ , then $Q = f_\mu(A)$, mod $OPS^{\mu-1}(M)$. An inductive argument easily gives the following.

Proposition 2.3. *Under hypothesis (2.4), given any $Q \in OPS^\mu(M)$, there exists $f \in S^\mu(\mathfrak{g}^*)$ such that $Q = f(A)$, mod $OPS^{-\infty}(M)$, and such that $\text{supp } \hat{f} \subset \mathcal{O}$.*

We now examine the commutator of an operator $f(A)$, given $f \in S^1(\mathfrak{g}^*)$, with an operator $P \in \mathcal{L}(L^2(M))$, assumed to be U -smooth. Using exponential coordinates, set

$$(2.5) \quad P(t) \sim \sum_{\beta \geq 0} P_\beta t^\beta, \quad P_\beta = D^\beta P(0)/\beta!,$$

where $P(t)$ is given by (2.1). It easily follows that each P_β is U -smooth, if P is. Using

$$(2.6) \quad U(t)P = P(t)U(t) \sim \sum_{\beta \geq 0} P_\beta t^\beta U(t),$$

and plugging into (2.2), we get

$$(2.7) \quad f(A)P \sim Pf(A) + \sum_{\beta > 0} P_\beta \int \hat{f}(t) t^\beta \kappa(t) U(t) dt \sim Pf(A) + \sum_{\beta > 0} P_\beta f_\beta(A).$$

Note that

$$f \in S^1(\mathfrak{g}^*) \Rightarrow f_\beta \in S^{1-|\beta|}(\mathfrak{g}^*) \Rightarrow f_\beta(A) \in OPS^{1-|\beta|}(M).$$

Since it is clear that each element of $OPS_{1,0}^0(M)$ is U -smooth, and the set of U -smooth operators forms an algebra, we deduce the following:

Proposition 2.4. *If P is U -smooth, then for all $f \in S^1(\mathfrak{g}^*)$, with $\hat{f}$ supported in $\mathcal{O}$,*

$$[f(A), P]: L^2(M) \rightarrow L^2(M),$$

and furthermore this commutator is U -smooth.

It follows that, for any $f_\nu \in S^1(\mathfrak{g}^*)$, with $\hat{f}_\nu$ supported in $\mathcal{O}$, the iterated commutators

$$\text{ad } f_N(A) \cdots \text{ad } f_1(A) \cdot P$$

are all bounded on $L^2(M)$, if P is U -smooth. In view of Proposition 2.3, the criterion of [B], [C] applies: for any $Q_\nu \in OPS^1(M)$,

$$\text{ad } Q_N \cdots \text{ad } Q_1 \cdot P: L^2(M) \rightarrow L^2(M).$$

This gives our main conclusion:

Theorem 2.5. *Under hypothesis (2.4), an operator $P \in \mathcal{L}(L^2(M))$ is in $OPS_{1,0}^0(M)$ if and only if it is U -smooth.*

We make some remarks on the hypothesis (2.4). Note that, for $(x, \xi) \in T^*M \setminus 0$,

$$(2.8) \quad \begin{aligned} D\Phi(x, \xi) \text{ is injective} &\Leftrightarrow \{da_j(x, \xi): 1 \leq j \leq m\} \text{ spans } T_{(x, \xi)} T^*M \\ &\Leftrightarrow \{H_{a_j}: 1 \leq j \leq m\} \text{ spans } T_{(x, \xi)} T^*M, \end{aligned}$$

where H_{a_j} are Hamiltonian vector fields, generating an associated action $U^\#$ of G on $T^*M \setminus 0$. This last condition holds provided

$$(2.9) \quad G \text{ acts transitively on } T^*M \setminus 0, \text{ via } U^\#;$$

in fact when $T^*M \setminus 0$ is connected these conditions are equivalent. We also note that the moment map is invariantly defined as

$$(2.10) \quad \Phi: T^*M \setminus 0 \rightarrow \mathfrak{g}^* \setminus 0;$$

see [GS]. Furthermore, Φ intertwines the $U^\#$ on $T^*M \setminus 0$ with the Ad^* action of G on $\mathfrak{g}^* \setminus 0$. The condition (2.9) implies that, under the image of Φ , $T^*M \setminus 0$ covers a coadjoint orbit of G in $\mathfrak{g}^* \setminus 0$. The hypothesis (2.4) would state that Φ maps $T^*M \setminus 0$ diffeomorphically onto such a coadjoint orbit.

An important example of this phenomenon is provided by the natural action of the conformal group $SO_e(n+1, 1)$ on the sphere S^n ,

$$(2.11) \quad U(g)v(x) = J_g(x)^{1/2}v(g^{-1} \cdot x),$$

where $J_g(x)$ is the usual Jacobian determinant. Thus Theorem 2.5 applies in this case. Thus we have proved the Theorem stated in the introduction.

We note that this U -action on $L^2(S^n)$ is the simplest sort of principal series representation of the Lie group $SO_e(n+1, 1)$. Given a general irreducible unitary representation U of a Lie group G on a Hilbert space H , one can consider the set of operators P on H such that the family (2.1) is smooth from G to $\mathcal{L}(H)$. As another sort of example, we mention the following result of [C]; see also [C2]. If G is the Heisenberg group (of dimension $2n+1$) and U the standard representation on $L^2(\mathbb{R}^n)$, then an operator P is U -smooth if and only if $P \in OPS_{0,0}^0(\mathbb{R}^n)$. It is of interest to consider families of U -smooth operators in other situations, but we will not pursue this here.

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