GENERALIZED NUMERICAL RANGES, JOINT POSITIVE DEFINITENESS AND MULTIPLE EIGENVALUES

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ABSTRACT. We prove a convexity theorem on a generalized numerical range that combines and generalizes the following results: 1) Friedland and Loewy's result on the existence of a nonzero matrix with multiple first eigenvalue in subspaces of hermitian matrices, 2) Bohnenblust's result on joint positive definiteness of hermitian matrices, 3) the Toeplitz-Hausdorff Theorem on the convexity of the classical numerical range and its various generalizations by Au-Yeung, Berger, Brickman, Halmos, Poon, Tsing and Westwick.

1. INTRODUCTION

The study of hermitian matrices arises naturally in many branches of mathematics. (See [HJ] for a nice introduction on the subject.) We are going to study the following areas of hermitian matrices: multiplicity of the largest eigenvalue, positive definiteness, convexity and inclusion of generalized numerical ranges.

Let $A_1, \dots, A_p$ be p $n \times n$ hermitian matrices and $\mathcal{K}$ the real linear subspace spanned by $A_1, \dots, A_p$. There has been considerable interest in studying the joint behavior of $A_1, \dots, A_p$. For example:

Friedland and Loewy [FL] proved that for every $1 < r < n$ there exists $k(r)$ such that if $\dim \mathcal{K} \geq k(r)$, then there exists a non-zero matrix A in $\mathcal{K}$ such that the largest eigenvalue of A has multiplicity $\geq r$. (See Proposition 1.3.)

Bohnenblust [Bo] gave a necessary and sufficient condition for the existence of a positive definite matrix in $\mathcal{K}$. (See Proposition 1.2.)

Au-Yeung and Poon [AP] showed that Bohnenblust's result is equivalent to a result which generalizes the Toeplitz-Hausdorff Theorem [To], [Hau] on the convexity of numerical range. (See Proposition 1.1.)

The Toeplitz-Hausdorff Theorem has been generalized in many directions by Au-Yeung [A1], Halmos [Hal1] and Berger [Be], Westwick [W], Au-Yeung and Tsing [AT1], [AT2]. Au-Yeung and Tsing [AT1], [AT2] have also shown that the convexity theorem of the joint c -numerical range is equivalent to some joint positive

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definiteness property, which, in turn, is equivalent to some inclusion relations on the joint $\mathbf{c}$ -numerical ranges.

In this paper, we are going to prove a theorem, which will combine and generalize all of the results mentioned above.

Let $\mathbf{R}$ be the field of real numbers, $\mathbf{C}$ the field of complex numbers and $\mathbf{Q}$ the skew field of real quaternions. We will use $\mathbf{F}$ to denote any one of $\mathbf{R}$, $\mathbf{C}$ or $\mathbf{Q}$. For an $n \times m$ matrix A over $\mathbf{F}$, A^* will denote the *conjugate transpose* of A . For this purpose, vectors in $\mathbf{F}^n$ will be identified with $1 \times n$ matrices over $\mathbf{F}$. $\mathcal{M}_n(\mathbf{F})$ will denote the set of all $n \times n$ matrices over $\mathbf{F}$. For $\mathbf{c} \in \mathbf{F}^n$, $[\mathbf{c}]$ will denote the diagonal matrix in $\mathcal{M}_n(\mathbf{F})$ with diagonal $\mathbf{c}$. Suppose $A_i \in \mathcal{M}_{n_i}(\mathbf{F})$, $1 \leq i \leq m$. Then $A_1 \oplus \cdots \oplus A_m$ will represent the block diagonal matrix formed by $A_1, \dots, A_m$.

Let $\mathcal{H}_n(\mathbf{F}) = \{A \in \mathcal{M}_n(\mathbf{F}) : A = A^*\}$ and $\mathcal{U}_n(\mathbf{F}) = \{A \in \mathcal{M}_n(\mathbf{F}) : AA^* = I_n\}$ be the *hermitian* and *unitary* matrices in $\mathcal{M}_n(\mathbf{F})$ respectively. Here, I_n denotes the $n \times n$ identity matrix. For each $n \geq 1$, $\mathcal{H}_n(\mathbf{F})$ is a real vector space of dimension $f_{\mathbf{F}}(n)$, where $f_{\mathbf{R}}(n) = n(n+1)/2$, $f_{\mathbf{C}}(n) = n^2$, and $f_{\mathbf{Q}}(n) = n(2n-1)$. Thus, we can identify $\mathcal{H}_n(\mathbf{F})$ with $\mathbf{R}^{f_{\mathbf{F}}(n)}$, equipped with the usual topology.

A subset S of $\mathbf{R}^m$ is said to be *convex* if for every $\mathbf{x}, \mathbf{y} \in S$, we have $t\mathbf{x} + (1-t)\mathbf{y} \in S$ for all $0 \leq t \leq 1$. Given a subset S of $\mathbf{R}^m$, $\text{conv } S$ will denote the smallest convex subset of $\mathbf{R}^m$ containing S . A subset $P \subseteq \mathbf{R}^m$ is called a *hyperplane* if there exist a subspace V of $\mathbf{R}^m$ and $a \in \mathbf{R}^m$ such that $P = \{a + v : v \in V\}$. A non-empty subset S of $\mathbf{R}^m$ is said to have a *convex boundary* if there exists a hyperplane P such that $S \subseteq P$ and the boundary of $\text{conv } S$, as a topological subspace of P , is contained in S .

Let $W^{\mathbf{F}}(A_1, A_2) = \{(\mathbf{x}A_1\mathbf{x}^*, \mathbf{x}A_2\mathbf{x}^*) : \mathbf{x} \in \mathbf{F}^n, \mathbf{x}\mathbf{x}^* = 1\}$, for $A_1, A_2 \in \mathcal{H}_n(\mathbf{F})$. The Toeplitz-Hausdorff Theorem [To], [Hau] states that $W^{\mathbf{C}}(A_1, A_2)$ is a convex subset of $\mathbf{R}^2$ for all $A_1, A_2 \in \mathcal{H}_n(\mathbf{C})$. Brickman [Br] proved that if $n > 2$, then $W^{\mathbf{R}}(A_1, A_2)$ is convex for all $A_1, A_2 \in \mathcal{H}_n(\mathbf{R})$. Au-Yeung [A1] gave a unified proof for the convexity of $W^{\mathbf{F}}(A_1, A_2)$ for $\mathbf{F} = \mathbf{R}$ ($n > 2$) and $\mathbf{F} = \mathbf{C}$ or $\mathbf{Q}$.

One direction of generalization of the Toeplitz-Hausdorff Theorem is to consider the set $W^{\mathbf{F}}(A_1, \dots, A_p) = \{(\mathbf{x}A_1\mathbf{x}^*, \dots, \mathbf{x}A_p\mathbf{x}^*) : \mathbf{x} \in \mathbf{F}^n, \mathbf{x}\mathbf{x}^* = 1\}$ for $A_1, \dots, A_p \in \mathcal{H}_n(\mathbf{F})$. Hausdorff [Hau] noted that $W^{\mathbf{C}}(A_1, A_2, A_3)$ has a convex boundary and remarked that $W^{\mathbf{C}}(A_1, A_2, A_3)$ is not convex in general. However, for $n \geq 3$, $W^{\mathbf{C}}(A_1, A_2, A_3)$ is convex for all $A_1, A_2, A_3 \in \mathcal{H}_n(\mathbf{C})$. This is a special case of the following

Proposition 1.1. *Let $\mathbf{F} = \mathbf{R}$, $\mathbf{C}$, or $\mathbf{Q}$, $1 \leq r \leq n-1$, $p < f_{\mathbf{F}}(r+1) - \delta_{n,r+1}$ and $A_1, \dots, A_p \in \mathcal{H}_n(\mathbf{F})$. Then the set*

$$W_r^{\mathbf{F}}(A_1, \dots, A_p) = \left\{ \left(\sum_{i=1}^r \mathbf{x}_i A_1 \mathbf{x}_i^*, \dots, \sum_{i=1}^r \mathbf{x}_i A_p \mathbf{x}_i^* \right) : \mathbf{x}_i \in \mathbf{F}^n, \sum_{i=1}^r \mathbf{x}_i \mathbf{x}_i^* = 1 \right\}$$

is convex. Here $\delta_{i,j}$ is the Kronecker delta.

Au-Yeung and Poon showed [AP] that Proposition 1.1 is equivalent to the following result of Bohnenblust [Bo], [Ta] on joint positive definiteness of hermitian matrices.

Proposition 1.2. *Let $\mathbf{F} = \mathbf{R}, \mathbf{C}$ or $\mathbf{Q}$, $1 \leq r \leq n-1$, $p < f_{\mathbf{F}}(r+1) - \delta_{n,r+1}$ and $A_1, \dots, A_p \in \mathcal{H}_n(\mathbf{F})$. Suppose $(0, \dots, 0) \notin W_r^{\mathbf{F}}(A_1, \dots, A_p)$. Then there exist $\alpha_1, \dots, \alpha_p \in \mathbf{R}$ such that the matrix $\sum_{i=1}^p \alpha_i A_i$ is positive definite.*

Bohnenblust proved Proposition 1.2 for $\mathbf{F} = \mathbf{R}$ and $\mathbf{C}$ but, as noted in [AP], the result remains valid for $\mathbf{F} = \mathbf{Q}$. Friedland and Lowey [FL] proved the next proposition and then showed that it is also equivalent to Proposition 1.2, for $\mathbf{F} = \mathbf{R}$ or $\mathbf{C}$.

Proposition 1.3. *Let $\mathbf{F} = \mathbf{R}$ or $\mathbf{C}$, $1 < r < n$, and $\mathbf{k}(r) = f_{\mathbf{F}}(n) - f_{\mathbf{F}}(n+1-r)$. Suppose $\mathcal{K}$ is a subspace of $\mathcal{H}_n(\mathbf{F})$ with dimension $\geq \mathbf{k}(r)$. Then there exists a non-zero matrix $A \in \mathcal{K}$ such that $\lambda_1(A) = \dots = \lambda_r(A)$.*

Here, for a matrix $B \in \mathcal{H}_n(\mathbf{F})$, $\lambda_1(B) \geq \lambda_2(B) \geq \dots \geq \lambda_n(B)$ are the eigenvalues of B arranged in decreasing order.

Remark 1.4. a) The bound $f_{\mathbf{F}}(r+1) - \delta_{n,r+1}$ in Proposition 1.1 is best possible in the sense that for every $1 \leq r \leq n-1$ and $p \geq f_{\mathbf{F}}(r+1) - \delta_{n,r+1}$, there exist $A_1, \dots, A_p \in \mathcal{H}_n(\mathbf{F})$ such that $W_r^{\mathbf{F}}(A_1, \dots, A_p)$ is not convex.

b) The set $W^{\mathbf{F}}(A_1, \dots, A_p)$ has appeared in the study of structured singular values in control theory [D], [FT], which leads to the problem of determining the smallest r such that every $\mathbf{w}$ in $\text{conv } W^{\mathbf{F}}(A_1, \dots, A_p)$ can be expressed as a convex combination of not more than r points in $W^{\mathbf{F}}(A_1, \dots, A_p)$. It has been shown in [P2] that the inequality $p < f_{\mathbf{F}}(r+1) - \delta_{n,r+1}$ gives the best possible bound for r .

Let $\mathbf{c} = (c_1, \dots, c_n) \in \mathbf{R}^n$, and $A_1, \dots, A_p \in \mathcal{H}_n(\mathbf{F})$. To state another generalization of the Toeplitz-Hausdorff Theorem, define

$$\begin{aligned} \mathcal{O}_n(\mathbf{F}) &= \{ \{ \mathbf{x}_1, \dots, \mathbf{x}_n \} : \mathbf{x}_i \in \mathbf{F}^n, \mathbf{x}_i \mathbf{x}_j^* = \delta_{ij} \}, \\ W_{\mathbf{c}}^{\mathbf{F}}(A_1, \dots, A_p) &= \left\{ \left(\sum_{i=1}^n c_i \mathbf{x}_i A_1 \mathbf{x}_i^*, \dots, \sum_{i=1}^n c_i \mathbf{x}_i A_p \mathbf{x}_i^* \right) : \{ \mathbf{x}_1, \dots, \mathbf{x}_n \} \in \mathcal{O}_n(\mathbf{F}) \right\}. \end{aligned}$$

Let $1 \leq k \leq n$. Halmos [Hal1] conjectured that if $\mathbf{c}$ has k coordinates equal to 1 and the rest 0, then $W_{\mathbf{c}}^{\mathbf{C}}(A_1, A_2)$ is convex. The conjecture was proven by Berger [Be], [Hal2]. Berger's result was generalized by Westwick [W], who proved the following result for $\mathbf{F} = \mathbf{C}$. A simple proof of Westwick's result is given in [P1]. The following proposition was proven by Au-Yeung and Tsing [AT1].

Proposition 1.5. *Let $\mathbf{c} \in \mathbf{R}^n$. If $\mathbf{F} = \mathbf{R}$ and $n > 2$ or $\mathbf{F} = \mathbf{C}, \mathbf{Q}$, then $W_{\mathbf{c}}^{\mathbf{F}}(A_1, A_2)$ is convex for all $A_1, A_2 \in \mathcal{H}_n(\mathbf{F})$.*

Au-Yeung [A2] has called for a unified proof of Propositions 1.1 and 1.5. The following result of Au-Yeung and Tsing [AT2] is the first step in this direction.

Proposition 1.6. *Let $n \geq 3$, $\mathbf{c} \in \mathbf{R}^n$. Then $W_{\mathbf{c}}^{\mathbf{C}}(A_1, A_2, A_3)$ is convex for all $A_1, A_2, A_3 \in \mathcal{H}_n(\mathbf{C})$.*

We note that for $\mathbf{c} = (1, 0, \dots, 0) \in \mathbf{R}^n$, $W_{\mathbf{c}}^{\mathbf{C}}(A_1, A_2, A_3) = W_1^{\mathbf{C}}(A_1, A_2, A_3)$. Thus, for $n \geq 3$ and $\mathbf{F} = \mathbf{C}$, Proposition 1.6 covers both Proposition 1.5 and the case when $r = 1$ in Proposition 1.1.

In this paper, we will prove a result which covers Propositions 1.1, 1.2, 1.3 1.5 and 1.6.

2. MAIN THEOREM

A square matrix with nonnegative entries is said to be **doubly stochastic** (d.s.) if every row sum and every column sum is equal to 1. Let $\Omega(n)$ be the set of all $n \times n$ d.s. matrices and $\Omega(n, r) = \{\oplus_{i=1}^m D_i : m \geq 1, D_i \in \Omega(n_i), n_i \leq r, \sum_{i=1}^m n_i = n\}$. We have $\Omega(n, 1) \subseteq \Omega(n, 2) \subseteq \cdots \subseteq \Omega(n, n) = \Omega(n)$.

For $\mathbf{c} = (c_1, \dots, c_n)$, let $c_{[1]} \geq \cdots \geq c_{[n]}$ be the decreasing rearrangement of the components of $\mathbf{c}$ and $\mathbf{c}_\downarrow = (c_{[1]}, \dots, c_{[n]})$.

Definition 2.1. Let $\mathbf{c} \in \mathbf{R}^n$, $\mathbf{F} = \mathbf{R}, \mathbf{C}$ or $\mathbf{Q}$ and $A_1, \dots, A_p \in \mathcal{H}_n(\mathbf{F})$. For $1 \leq r \leq n$, define

$$\mathcal{C}(\mathbf{c}, r) = \{\mathbf{c}_\downarrow D : D \in \Omega(n, r)\}, \text{ and}$$

$$W_{\mathbf{c}, r}^{\mathbf{F}}(A_1, \dots, A_p) = \{(tr [\mathbf{b}] U A_1 U^*, \dots, tr [\mathbf{b}] U A_p U^*) : \mathbf{b} \in \mathcal{C}(\mathbf{c}, r), U \in \mathcal{U}_n(\mathbf{F})\},$$

where $tr A$ denotes the trace of A .

The main result in this paper is

Theorem 2.2. Let $\mathbf{c} \in \mathbf{R}^n$, $\mathbf{F} = \mathbf{R}, \mathbf{C}$ or $\mathbf{Q}$, $1 \leq r \leq n-1$, $p < f_{\mathbf{F}}(r+1) - \delta_{n, r+1}$ and $A_1, \dots, A_p \in \mathcal{H}_n(\mathbf{F})$. Then $W_{\mathbf{c}, r}^{\mathbf{F}}(A_1, \dots, A_p)$ is convex.

Remark 2.3. We note that for $\mathbf{e} = (1, 0, \dots, 0)$ and $\mathbf{c} \in \mathbf{R}^n$, we have

$$W_{\mathbf{e}, r}^{\mathbf{F}}(A_1, \dots, A_p) = W_r^{\mathbf{F}}(A_1, \dots, A_p), \text{ and}$$

$$W_{\mathbf{e}, 1}^{\mathbf{F}}(A_1, \dots, A_p) = W_{\mathbf{c}}^{\mathbf{F}}(A_1, \dots, A_p).$$

For $r = 1$, $f_{\mathbf{F}}(r+1) - \delta_{n, r+1} > \begin{cases} 2 & \text{if } \mathbf{F} = \mathbf{R} \text{ and } n \geq 3 \text{ or } \mathbf{F} = \mathbf{C}, \mathbf{Q} \\ 3 & \text{if } n \geq 3 \text{ and } \mathbf{F} = \mathbf{C}, \mathbf{Q} \end{cases}$. Therefore,

Theorem 2.2 is a generalization of Propositions 1.1, 1.5 and 1.6.

For the rest of this section, $\mathbf{F} = \mathbf{R}, \mathbf{C}$ or $\mathbf{Q}$. If $\mathbf{d}^i \in \mathbf{R}^{n_i}$, for $1 \leq i \leq m$ with $\sum_{i=1}^m n_i = n$, then $(\mathbf{d}^1, \mathbf{d}^2, \dots, \mathbf{d}^m)$ will denote the vector in $\mathbf{R}^n$ formed by the juxtaposition of the components in $\mathbf{d}^i$.

Lemma 2.4. a) Let $A \in \mathcal{M}_n(\mathbf{F})$. Then $A \in \mathcal{H}_n(\mathbf{F})$ if and only if there exist $U \in \mathcal{U}_n(\mathbf{F})$ and $\mathbf{c} \in \mathbf{R}^n$ such that $A = U[\mathbf{c}]U^*$.

b) Let $U \in \mathcal{U}_n(\mathbf{R})$. There is a path in $\mathcal{U}_n(\mathbf{R})$ connecting U to I_n if and only if $\det U = 1$. For $\mathbf{F} = \mathbf{C}$ or $\mathbf{Q}$, $\mathcal{U}_n(\mathbf{F})$ is path connected.

Proof. For $\mathbf{F} = \mathbf{R}$ or $\mathbf{C}$, the result is well known [HJ]. For $\mathbf{F} = \mathbf{Q}$, a) and b) follow from Theorem 9 and 8 of Lee [Le] respectively. $\square$

For $A \in \mathcal{M}_n(\mathbf{F})$, let $Tr A = \text{Re } tr A$, the real part of $tr A$. Then we have

Lemma 2.5. For $A, B \in \mathcal{M}_n(\mathbf{F})$, $\langle A, B \rangle = Tr AB^*$ defines a real valued inner product on $\mathcal{M}_n(\mathbf{F})$. Moreover, $Tr AB = Tr BA$ for all $A, B \in \mathcal{M}_n(\mathbf{F})$.

Proof. Since the result is well known for $\mathbf{F} = \mathbf{R}$ or $\mathbf{C}$, it remains to prove the case when $\mathbf{F} = \mathbf{Q}$. Recall that $\mathbf{Q}$ is generated by $\mathbf{C}$ and $\mathbf{j}$ such that $a\mathbf{j} = \mathbf{j}\bar{a}$ for all $a \in \mathbf{C}$. Thus, every $A \in \mathcal{M}_n(\mathbf{Q})$ can be expressed uniquely as $A = A_1 + \mathbf{j}A_2$, with $A_1, A_2 \in \mathcal{M}_n(\mathbf{C})$. This induces (see Lee [Le]) a unital real algebra isomorphism f of $\mathcal{M}_n(\mathbf{Q})$ into a subalgebra of $\mathcal{M}_{2n}(\mathbf{C})$ by

$$f(A_1 + \mathbf{j}A_2) = \begin{bmatrix} A_1 & -\overline{A_2} \\ A_2 & \overline{A_1} \end{bmatrix}.$$

Here, $\bar{A}$ denotes the *conjugate* of A . Since $\text{Tr } A = (\text{tr } f(A)) / 2$, the result follows from the case when $\mathbf{F} = \mathbf{C}$. $\square$

Let $\mathbf{b}, \mathbf{c} \in \mathbf{R}^n$. We will write $\mathbf{b} \prec \mathbf{c}$ if $\mathbf{b} \in \mathcal{C}(\mathbf{c}, n)$. By the Hardy-Littlewood-Polya Theorem [HLP], [MO], we have $\mathbf{b} \prec \mathbf{c}$ if and only if

$$(1) \quad \sum_{i=1}^k b_{[i]} \leq \sum_{i=1}^k c_{[i]} \quad (k = 1, \dots, n-1) \text{ and } \sum_{i=1}^n b_{[i]} = \sum_{i=1}^n c_{[i]}.$$

Lemma 2.6. For $\mathbf{c} \in \mathbf{R}^n$, let $\mathcal{U}_{\mathbf{F}}(\mathbf{c}, r) = \{U^*[\mathbf{b}]U : \mathbf{b} \in \mathcal{C}(\mathbf{c}, r), U \in \mathcal{U}_n(\mathbf{F})\}$. We have

- a) For $1 \leq r \leq n$, $\text{conv } \mathcal{U}_{\mathbf{F}}(\mathbf{c}, r) = \{U^*[\mathbf{b}]U : \mathbf{b} \prec \mathbf{c}, U \in \mathcal{U}_n(\mathbf{F})\} = \mathcal{U}_{\mathbf{F}}(\mathbf{c}, n)$.
- b) $\{\mathbf{b} : \mathbf{b} \prec \mathbf{c}\} = \{\text{diag } U[\mathbf{c}]U^* : U \in \mathcal{U}_n(\mathbf{F})\}$
 $= \{\text{diag } U[\mathbf{b}]U^* : U \in \mathcal{U}_n(\mathbf{F}), \mathbf{b} \prec \mathbf{c}\}.$
- c) $\mathcal{U}_{\mathbf{F}}(\mathbf{c}, n-1)$ has a convex boundary.

Proof. We may assume that $\mathbf{c} = \mathbf{c}_{\downarrow}$.

- a) Since $\mathcal{U}_{\mathbf{F}}(\mathbf{c}, 1) \subseteq \dots \subseteq \mathcal{U}_{\mathbf{F}}(\mathbf{c}, n)$, it suffices to prove that $\text{conv } \mathcal{U}_{\mathbf{F}}(\mathbf{c}, 1) = \mathcal{U}_{\mathbf{F}}(\mathbf{c}, n)$. Thompson [Th]¹ proved the result for $\mathbf{F} = \mathbf{C}$ and it was pointed out by Au-Yeung and Tsing [AT1] that the result also holds for $\mathbf{F} = \mathbf{R}$ and $\mathbf{Q}$.
- b) The first equality is due to Horn [Ho] and Au-Yeung and Tsing [AT1]. The second equality follows from the first equality.
- c) Let $P_{\mathbf{c}} = \{A \in \mathcal{H}_n(\mathbf{F}) : \text{Tr } A = \sum_{i=1}^n c_i\} = \{[\mathbf{c}] + B : B \in \mathcal{H}_n(\mathbf{F}), \text{tr } B = 0\}$. Then $P_{\mathbf{c}}$ is a hyperplane containing $\mathcal{U}_{\mathbf{F}}(\mathbf{c}, n)$. From a), $\text{conv } \mathcal{U}_{\mathbf{F}}(\mathbf{c}, n-1) = \mathcal{U}_{\mathbf{F}}(\mathbf{c}, n) \subseteq P_{\mathbf{c}}$. Suppose $B \in \mathcal{U}_{\mathbf{F}}(\mathbf{c}, n)$. Then there exists $U \in \mathcal{U}_n(\mathbf{F})$ and $\mathbf{b} \in \mathcal{C}(\mathbf{c}, n)$ such that $B = U[\mathbf{b}]U^*$. If equality holds in (1) for some $1 \leq k < n$, then $B \in \mathcal{U}_{\mathbf{F}}(\mathbf{c}, n-1)$. Suppose strict inequality holds in (1) for all $1 \leq k < n$. Let $A \in P_{\mathbf{c}}$ have eigenvalues $\lambda_1 \geq \dots \geq \lambda_n$. If A is sufficiently close to B in $P_{\mathbf{c}}$, then we have $\sum_{i=1}^k \lambda_i < \sum_{i=1}^k c_i$ for $k = 1, \dots, n-1$. Hence, $A \in \mathcal{U}_{\mathbf{F}}(\mathbf{c}, n)$. Therefore the boundary of the convex set $\mathcal{U}_{\mathbf{F}}(\mathbf{c}, n)$ in $P_{\mathbf{c}}$ is contained in $\mathcal{U}_{\mathbf{F}}(\mathbf{c}, n-1)$. $\square$

Remark 2.7. Let $\mathbf{b} \in \mathbf{R}^n$, $A \in \mathcal{H}_n(\mathbf{F})$ and $U \in \mathcal{U}_n(\mathbf{F})$. It follows from Lemma 2.5 and 2.6 b) that $\text{tr } [\mathbf{b}]U A U^* = \text{Tr } [\mathbf{b}]U A U^* = \text{Tr } U^*[\mathbf{b}]U A$. Thus we have

$$W_{\mathbf{c}, r}^{\mathbf{F}}(A_1, \dots, A_p) = \{(\text{Tr } B A_1, \dots, \text{Tr } B A_p) : B \in \mathcal{U}_{\mathbf{F}}(\mathbf{c}, r)\}, \text{ and}$$

$$\text{conv } W_{\mathbf{c}, r}^{\mathbf{F}}(A_1, \dots, A_p) = W_{\mathbf{c}, n}^{\mathbf{F}}(A_1, \dots, A_p) \text{ for all } 1 \leq r \leq n.$$

First, we prove a special case of Theorem 2.2.

Lemma 2.8. Let $p \leq f_{\mathbf{F}}(n) - 2$. Then $W_{\mathbf{c}, n-1}^{\mathbf{F}}(A_1, \dots, A_p)$ is convex for all $A_1, \dots, A_p \in \mathcal{H}_n(\mathbf{F})$.

Proof. Let $\mathbf{w} = (w_1, \dots, w_p) \in \text{conv } W_{\mathbf{c}, n-1}^{\mathbf{F}}(A_1, \dots, A_p) = W_{\mathbf{c}, n}^{\mathbf{F}}(A_1, \dots, A_p)$. Then, by Remark 2.7, there exists $B \in \mathcal{U}_{\mathbf{F}}(\mathbf{c}, n)$ such that $w_j = \text{Tr } B A_j$ for all $1 \leq j \leq p$. Since $p \leq f_{\mathbf{F}}(n) - 2$, there exists a non-zero $X \in \mathcal{H}_n(\mathbf{F})$ such that $\text{Tr } X = \text{Tr } X A_j = 0$ for all $1 \leq j \leq p$. For all $t \in \mathbf{R}$, we have $B + tX \in P_{\mathbf{c}}$ (as defined in the proof of Lemma 2.6 c)) and $\text{Tr } (B + tX) A_j = w_j$ for all $1 \leq j \leq p$. Since $\mathcal{U}_{\mathbf{F}}(\mathbf{c}, n)$ is bounded, we have, by Lemma 2.6 c), $B + tX \in \mathcal{U}_{\mathbf{F}}(\mathbf{c}, n-1)$ for some t . $\square$

¹The author is thankful to Professor Tin Yau Tam for pointing out this reference.

Lemma 2.9. *Let $p = f_{\mathbf{F}}(n) - 1$ and $\mathcal{K}$ be the subspace spanned by $A_1, \dots, A_p$ in $\mathcal{H}_n(\mathbf{F})$. Then we have*

- a) *If $I_n \in \mathcal{K}$ or $\dim \mathcal{K} < p$, then $W_{\mathbf{c}, n-1}^{\mathbf{F}}(A_1, \dots, A_p)$ is convex.*
- b) *$W_{\mathbf{c}, n-1}^{\mathbf{F}}(A_1, \dots, A_p)$ has a convex boundary.*

Proof. a) Suppose $B_1, \dots, B_q$ is a basis of $\mathcal{K}$. Then $W_{\mathbf{c}, n-1}^{\mathbf{F}}(A_1, \dots, A_p)$ is convex if and only if $W_{\mathbf{c}, n-1}^{\mathbf{F}}(B_1, \dots, B_q)$ is convex. If $q < p$, the result follows from Lemma 2.8.

Suppose $I_n \in \mathcal{K}$ and $q = p$. Then $A_1, \dots, A_p$ is a basis of $\mathcal{K}$. We may assume that $A_p = I_n$. Since $\text{Tr } X I_n = \sum_{i=1}^n c_i = s$ is independent of $X \in \mathcal{U}_{\mathbf{F}}(\mathbf{c}, n-1)$, we have $W_{\mathbf{c}, n-1}^{\mathbf{F}}(A_1, \dots, A_p) = \{(\mathbf{w}, s) : \mathbf{w} \in W_{\mathbf{c}, n-1}^{\mathbf{F}}(A_1, \dots, A_{p-1})\}$. Hence, the convexity of $W_{\mathbf{c}, n-1}^{\mathbf{F}}(A_1, \dots, A_p)$ follows from that of $W_{\mathbf{c}, n-1}^{\mathbf{F}}(A_1, \dots, A_{p-1})$.

b) From a), we only need to consider the case when $\dim \mathcal{K} = p$ and $I_n \notin \mathcal{K}$; then $\{I_n, A_1, \dots, A_p\}$ is a basis for $\mathcal{H}_n(\mathbf{F})$. We have

$$W_{\mathbf{c}, n-1}^{\mathbf{F}}(A_1, \dots, A_p) \cong W_{\mathbf{c}, n-1}^{\mathbf{F}}(A_1, \dots, A_p, I_n) \cong \mathcal{U}_{\mathbf{F}}(\mathbf{c}, n-1).$$

Hence, by Lemma 2.6 c), $W_{\mathbf{c}, n-1}^{\mathbf{F}}(A_1, \dots, A_p)$ has a convex boundary. $\square$

Let $\mathbf{b} \in \mathbf{R}^n$ and $1 \leq k \leq \ell \leq n$; we will denote $(b_k, \dots, b_\ell)$ by $\mathbf{b}[k, \ell]$. Similarly, for $A = (a_{i,j}) \in \mathcal{M}_n(\mathbf{F})$, $A[k, \ell]$ will denote the principal submatrix $(a_{i,j})_{k \leq i, j \leq \ell}$ of A .

Lemma 2.10. *Let $\mathcal{K}$ be a subspace of $\mathcal{H}_n(\mathbf{F})$ of dimension $f_{\mathbf{F}}(n-1) - 1$. Then there exist a nonzero $A \in \mathcal{K}$ and $U \in \mathcal{U}_n(\mathbf{F})$ such that $(UAU^*)[1, n-1] = \alpha I_{n-1}$ for some $\alpha \in \mathbf{R}$. Moreover, U can be chosen in the path connected component of I_n in $\mathcal{U}_n(\mathbf{F})$.*

Proof. If $I_n \in \mathcal{K}$, then we are done. Suppose $I_n \notin \mathcal{K}$. Let $\mathcal{K}_1$ be the subspace spanned by I_n and $\mathcal{K}$. Let $\mathcal{L} = \mathcal{K}_1^\perp$ be the orthogonal complement of $\mathcal{K}_1$ in $\mathcal{H}_n(\mathbf{F})$, with respect to the inner product defined in Lemma 2.5. Let $m = \dim \mathcal{L} = f_{\mathbf{F}}(n) - f_{\mathbf{F}}(n-1)$. By a result of Adams, Lax and Phillips ([ALP, Theorem 1]), there exists a nonzero singular matrix B in $\mathcal{L}$. Choose $U \in \mathcal{U}_n(\mathbf{F})$ such that the last row and last column of UBU^* are zero. Let $\mathcal{D} = \{D \in \mathcal{H}_n(\mathbf{F}) : D[1, n-1] = 0\}$. Since $UBU^* \in \mathcal{D}^\perp \cap (U\mathcal{L}U^*)$ and $\dim U\mathcal{L}U^* = \dim \mathcal{L} = \dim \mathcal{D}$, we have

$$\begin{aligned} & \dim (\mathcal{D} \cap (U\mathcal{L}U^*)^\perp) \\ &= \dim \mathcal{D} + \dim (U\mathcal{L}U^*)^\perp - \dim (D + (U\mathcal{L}U^*)^\perp) \\ &= f_{\mathbf{F}}(n) - \dim (D + (U\mathcal{L}U^*)^\perp) \\ &= \dim \left((D + (U\mathcal{L}U^*)^\perp)^\perp \right) = \dim (\mathcal{D}^\perp \cap (U\mathcal{L}U^*)) > 0. \end{aligned}$$

Let D be a nonzero matrix in $\mathcal{D} \cap (U\mathcal{L}U^*)^\perp = \mathcal{D} \cap U\mathcal{K}_1U^*$. Then there exist $A \in \mathcal{K}$ and $\alpha \in \mathbf{R}$ such that $U(A - \alpha I_n)U^* = D \in \mathcal{D}$. Thus, $(UAU^*)[1, n-1] = \alpha I_{n-1}$. If $A = 0$, then $\alpha = 0$ because $D[1, n-1] = 0$. This would imply $D = 0$, a contradiction. Hence, we have $A \neq 0$.

By Lemma 2.4 b), U is in the connected component of I_n in $\mathcal{U}_n(\mathbf{F})$ for $\mathbf{F} = \mathbf{C}$ and $\mathbf{Q}$. For $\mathbf{F} = \mathbf{R}$, by multiplying U by $I_{n-1} \oplus [-1]$, if necessary, we get $\det U = 1$ and the result follows. $\square$

Remark 2.11. Suppose $A \in \mathcal{H}_n(\mathbf{F})$ has eigenvalues $\lambda_1 \geq \dots \geq \lambda_n$ and $\alpha \in \mathbf{R}$. Then direct computation shows that $\lambda_2 = \dots = \lambda_{n-1} = \alpha$ if and only if there

exists $U \in \mathcal{U}_n(\mathbf{F})$ such that $UAU^*[1, n-1] = \alpha I_{n-1}$. Therefore Lemma 2.10 can be restated as follows.

Proposition 2.12. *Let $\mathcal{K}$ be a subspace of $\mathcal{H}_n(\mathbf{F})$ of dimension $f_{\mathbf{F}}(n-1) - 1$. Then there exists a nonzero $A \in \mathcal{K}$ such that $\lambda_2(A) = \cdots = \lambda_{n-1}(A)$.*

It would be interesting to see if there is any direct relationship between Propositions 1.3 and 2.12.

The following lemma is an extension of an idea contained in a paper by Au-Yeung and Tsing [AT2].

Lemma 2.13. *Let $r = n - 2 \geq 1$, $p = f_{\mathbf{F}}(r+1) - 1$ and $A_1, \dots, A_p \in \mathcal{H}_n(\mathbf{F})$. Suppose $\mathbf{c} = \mathbf{c}_1 \in \mathbf{R}^{n-1}$, $\mathbf{b} \in \mathcal{C}(\mathbf{c}, n-1)$ and $e \in \mathbf{R}$ such that either $e \leq c_{n-1}$ or $e \geq c_1$. Then there exist $\mathbf{d} \in \mathcal{C}(\mathbf{c}, r)$ and $U \in \mathcal{U}_n(\mathbf{F})$ such that $\text{Tr}([\mathbf{b}] \oplus [e])A_j = \text{Tr}([\mathbf{d}] \oplus [e])UA_jU^*$ for all $1 \leq j \leq p$.*

Proof. Let $\mathcal{K}$ be the subspace spanned by $A_1, \dots, A_p$. If $\dim \mathcal{K} < p$, let $B_i = A_i[1, n-1]$ for $1 \leq i \leq p$. Then, by Lemma 2.9 a), $W_{\mathbf{c}, r}^{\mathbf{F}}(B_1, \dots, B_p)$ is convex and the result follows.

Suppose $\dim \mathcal{K} = p$. Then by Lemma 2.10, we can get U_1 in the path connected component of I_n in $\mathcal{U}_n(\mathbf{F})$ such that $(U_1AU_1^*)[1, n-1] = \alpha I_{n-1}$ for some nonzero $A \in \mathcal{K}$ and $\alpha \in \mathbf{R}$.

Let $U(t)$, $0 \leq t \leq 1$, be a path in $\mathcal{U}_n(\mathbf{F})$ such that $U(0) = I_n$ and $U(1) = U_1$. For every $0 \leq t \leq 1$ and $1 \leq j \leq p$, let $(U(t)A_jU(t)^*)[1, n-1] = B_j(t)$ and $(U(t)A_jU(t)^*)[n, n] = a_j(t)$. Define

$$W(t) = W_{\mathbf{c}, r}^{\mathbf{F}}(B_1(t), \dots, B_p(t)) + \{e(a_1(t), \dots, a_p(t))\}.$$

Then $W(t)$ varies continuously with t . Let $\mathcal{K}_1$ be the subspace of $\mathcal{H}_{n-1}(\mathbf{F})$ spanned by $B_1(1), \dots, B_p(1)$. Then either $I_{n-1} \in \mathcal{K}_1$ or $\dim \mathcal{K}_1 < p$. We have

- a) $\mathbf{q} = (\text{Tr}([\mathbf{b}] \oplus [e])A_j)_{1 \leq j \leq p} \in \text{conv } W(0)$.
- b) For each $0 \leq t \leq 1$, $W(t)$ has a convex boundary and is contained in

$$\left\{ (\text{Tr}([\mathbf{d}] \oplus [e])UA_jU^*)_{1 \leq j \leq p} : \mathbf{d} \in \mathcal{C}(\mathbf{c}, r), U \in \mathcal{U}_n(\mathbf{F}) \right\}.$$

- c) $W(1)$ is convex.

If $\mathbf{q} \in W(0)$, then the result follows from b). Suppose $W(0)$ is not convex and $\mathbf{q}$ lies in the interior of $\text{conv } W(0)$.

Since $W(t)$ varies continuously with t and each $W(t)$ has a convex boundary, we have either 1) $W(t)$ is convex for some $t > 0$ and $\mathbf{q} \in W(t)$ or 2) there exists $t < 1$ such that $\mathbf{q}$ lies on the boundary of $\text{conv } W(t)$. In both cases, the result follows from b). $\square$

Proof of Theorem 2.2. For $r = n - 1$, the result follows from Lemma 2.8. So, we may assume that $1 \leq r \leq n - 2$. Therefore, $2 \leq r+1 \leq n-1$ and $p \leq f_{\mathbf{F}}(r+1) - 1$.

Suppose $\mathbf{w} \in W_{\mathbf{c}, n}^{\mathbf{F}}(A_1, \dots, A_p)$. Let $s = \min\{i : \mathbf{w} \in W_{\mathbf{c}, i}^{\mathbf{F}}(A_1, \dots, A_p)\}$. It suffices to show that $s \leq r$. Let $\mathbf{w} = (w_1, \dots, w_p)$. There exist $\mathbf{b} \in \mathcal{C}(\mathbf{c}, s)$ and $U \in \mathcal{U}_n(\mathbf{F})$ such that $w_j = \text{Tr}[\mathbf{b}]UA_jU^*$ for all $1 \leq j \leq p$. Moreover, there exist $D_i \in \Omega(n_i)$ with $n_i \leq s$ for $1 \leq i \leq m$ and $\sum_{i=1}^m n_i = n$ such that $\mathbf{b} = \mathbf{c}(\oplus_{i=1}^m D_i)$. Let $k_0 = 0$, $k_i = \sum_{j=1}^i n_j$, $\mathbf{c}^i = \mathbf{c}[k_{i-1}+1, k_i]$ and $\mathbf{b}^i = \mathbf{b}[k_{i-1}+1, k_i]$ for $1 \leq i \leq m$. Let $\mathcal{A}_j^i = (UA_jU^*)[k_{i-1}+1, k_i]$ for $1 \leq j \leq p$. Then $w_j = \sum_{i=1}^m \text{Tr}[\mathbf{b}^i]\mathcal{A}_j^i$, for all $1 \leq j \leq p$.

If $n_i > r+1$ for some i , then it follows from $p \leq f_{\mathbf{F}}(r+2)-2$ and Lemma 2.8 that there exist $\mathbf{d}^i \in \mathcal{C}(\mathbf{c}^i, r+1)$ and $U_i \in \mathcal{U}_{n_i}(\mathbf{F})$ such that $\text{Tr}[\mathbf{b}^i]\mathcal{A}_j^i = \text{Tr}[\mathbf{d}^i]U_i\mathcal{A}_j^iU_i^*$ for all $1 \leq j \leq p$. Hence we may assume that $s \leq r+1$. Suppose $n_i = r+1$. If $p \leq f_{\mathbf{F}}(r+1)-2$, then by Lemma 2.8, $W_{\mathbf{c}^i, n_i}^{\mathbf{F}}(\mathcal{A}_1^i, \dots, \mathcal{A}_p^i)$ is convex. If $p = f_{\mathbf{F}}(r+1)-1$, then we can apply Lemma 2.13 to $\mathbf{b}^i, \mathbf{c}^i$ and $(UAU^*)[k_{i-1}+1, k_i+1]$ if $i < m$ (or $(UAU^*)[k_{i-1}, k_i]$ if $i = m$). In all cases, we can get $\mathbf{d}^i \in \mathcal{C}(\mathbf{c}^i, r)$, $\mathbf{d} = (\mathbf{b}^1, \dots, \mathbf{b}^{i-1}, \mathbf{d}^i, \mathbf{b}^{i+1}, \dots, \mathbf{b}^m)$ and $V \in \mathcal{U}_n(\mathbf{F})$ such that $w_j = \text{Tr}[\mathbf{d}]V\mathcal{A}_jV^*$ for all $1 \leq j \leq p$. Repeating this procedure, we get $s \leq r$. $\square$

3. SOME RELATED RESULTS

Proposition 3.1. *Let $\mathbf{F} = \mathbf{R}, \mathbf{C}$ or $\mathbf{Q}$ and $A_1, \dots, A_p \in \mathcal{H}_n(\mathbf{F})$. Then for every $\mathbf{c} = (c_1, \dots, c_n) \in \mathbf{R}^n$ and $1 \leq r \leq n$, the following conditions are equivalent:*

- a) $W_{\mathbf{c}, r}^{\mathbf{F}}(A_1, \dots, A_p)$ is convex.
- b) If $(r_1, \dots, r_p) \in \mathbf{R}^p \setminus W_{\mathbf{c}, r}^{\mathbf{F}}(A_1, \dots, A_p)$, then there exist $\alpha_1, \dots, \alpha_p \in \mathbf{R}$ such that

$$\text{Tr } X \left(\sum_{j=1}^p \alpha_j A_j \right) > \sum_{j=1}^p \alpha_j r_j, \text{ for all } X \in \mathcal{U}_{\mathbf{F}}(\mathbf{c}, n).$$

- c) If $(r_1, \dots, r_p) \in \mathbf{R}^p \setminus W_{\mathbf{c}, r}^{\mathbf{F}}(A_1, \dots, A_p)$, then there exist $\alpha_1, \dots, \alpha_p \in \mathbf{R}$ such that the eigenvalues of $A = \sum_{j=1}^p \alpha_j A_j$ satisfies $\sum_{i=1}^n c_{\sigma(i)} \lambda_i(A) > \sum_{j=1}^p \alpha_j r_j$, for all permutation σ of $\{1, \dots, n\}$.
- d) $W_{\mathbf{b}, r}^{\mathbf{F}}(A_1, \dots, A_p) \subseteq W_{\mathbf{c}, r}^{\mathbf{F}}(A_1, \dots, A_p)$ for all $\mathbf{b} \prec \mathbf{c}$.

Some special cases of Proposition 3.1 are given by Au-Yeung and Tsing in [AT1] (for $p = 2$ and $r = 1$) and [AT2] (for $\mathbf{F} = \mathbf{C}$ and $r = 1$). The proof of the general case is similar to the one given in [AT1]. Also, see [A2] for some related results.

Suppose $1 \leq r \leq n-1$ and $p < f_{\mathbf{F}}(r+1) - \delta_{n, r+1}$. It follows from Theorem 2.2 that all the conditions in Proposition 3.1 hold. In particular, by putting $(r_1, \dots, r_p) = (0, \dots, 0)$ in Proposition 3.1 c), we have the following generalization of Proposition 1.2.

Theorem 3.2. *Let $\mathbf{F} = \mathbf{R}, \mathbf{C}$ or $\mathbf{Q}$, $1 \leq r \leq n-1$, $p < f_{\mathbf{F}}(r+1) - \delta_{n, r+1}$ and $A_1, \dots, A_p \in \mathcal{H}_n(\mathbf{F})$. Suppose $(0, \dots, 0) \notin W_{\mathbf{c}, r}^{\mathbf{F}}(A_1, \dots, A_p)$. Then there exist $\alpha_1, \dots, \alpha_p \in \mathbf{R}$ such that the eigenvalues of $A = \sum_{j=1}^p \alpha_j A_j$ satisfy $\sum_{i=1}^n c_{\sigma(i)} \lambda_i(A) > 0$ for every permutation σ of $\{1, \dots, n\}$.*

From Theorem 3.2, we have the following generalization of Proposition 1.3.

Theorem 3.3. *Let $\mathbf{F} = \mathbf{R}, \mathbf{C}$ or $\mathbf{Q}$, $1 < r < n$, and $\mathbf{k}(r) = f_{\mathbf{F}}(n) - f_{\mathbf{F}}(n+1-r)$. Suppose $\mathcal{K}$ be a subspace of $\mathcal{H}_n(\mathbf{F})$ with dimension $\geq \mathbf{k}(r)$. Then for every $\mathbf{c} \in \mathbf{R}^n$, with $c_i \neq c_j$ for some i, j , there exists a nonzero matrix $A \in \mathcal{K}$ such that $A = \lambda I + tB$ for some $\lambda, t \in \mathbf{R}$ and $B \in \mathcal{U}_{\mathbf{F}}(\mathbf{c}, n-r)$.*

Proof. Without loss of generality, we may assume $c_1 \geq \dots \geq c_n > 0$. Let $\mathcal{K}$ be a subspace of $\mathcal{H}_n(\mathbf{F})$ with dimension $\geq \mathbf{k}(r) = f_{\mathbf{F}}(n) - f_{\mathbf{F}}(n+1-r)$. Suppose the contrary that for every nonzero $A \in \mathcal{K}$, we have $A \neq \lambda I_n + tB$ for all $\lambda, t \in \mathbf{R}$ and $B \in \mathcal{U}_{\mathbf{F}}(\mathbf{c}, n-r)$. Then 1) $I_n \notin \mathcal{K}$ and 2) $\mathcal{K}_1 \cap \mathcal{U}_{\mathbf{F}}(\mathbf{c}, n-r) = \emptyset$, where $\mathcal{K}_1$ is the subspace spanned by $\mathcal{K}$ and I_n . Let $\mathcal{L} = \mathcal{K}_1^\perp$. It follows from 1) that

$\dim \mathcal{L} = p < f_{\mathbf{F}}(n+1-r)$. Since $1 < r < n$, we have $0 < n-r < n-1$. Let $\{A_1, \dots, A_p\}$ be a basis of $\mathcal{L}$. 2) implies that $(0, \dots, 0) \notin W_{\mathbf{c}, n-r}^{\mathbf{F}}(A_1, \dots, A_p)$. Hence, by Theorem 3.2, there exists $A \in \mathcal{L}$ such that $\sum_{i=1}^n c_{\sigma(i)} \lambda_i(A) > 0$ for every permutation σ of $\{1, \dots, n\}$. This implies that $\text{Tr } A = \sum_{i=1}^n \lambda_i(A) > 0$ because $\sum_{i=1}^n c_i > 0$. But $I_n \in \mathcal{K}_1 = \mathcal{L}^\perp$. So we have $\text{Tr } A = \text{Tr } A I_n = 0$, a contradiction. $\square$

As we have pointed out in Remark 1.4 a), the bound $f_{\mathbf{F}}(r+1) - \delta_{n,r+1}$ is best possible for Proposition 1.1. The following theorem is an extension of this result.

Theorem 3.4. *Suppose $c_1 \geq \dots \geq c_n$. If $c_1 > c_{r+1}$ or $c_{n-r} > c_n$ and $p \geq f_{\mathbf{F}}(r+1) - \delta_{n,r+1}$, then there exist $A_1, \dots, A_p \in \mathcal{H}_n(\mathbf{F})$ such that $W_{\mathbf{c}, r}^{\mathbf{F}}(A_1, \dots, A_p)$ is not convex.*

Proof. It suffices to prove the case when $p = f_{\mathbf{F}}(r+1) - \delta_{n,r+1}$.

We may assume that $c_1 > c_{r+1} \geq c_n > 0$. Let $d = \left(\sum_{i=1}^{r+1} c_i\right)(r+1)^{-1}$ and $s = \sum_{i=1}^n c_i$. Let $\mathcal{K} = \{A \in \mathcal{H}_n(\mathbf{F}) : A[1, r+1] = \lambda I_{r+1} \text{ and } d \text{tr } A = \lambda s\}$. Then $\mathcal{K}$ has dimension $f_{\mathbf{F}}(n) - p$. Let $A_1, \dots, A_p \in \mathcal{H}_n(\mathbf{F})$ be a basis of $\mathcal{K}^\perp$. Since $d I_{r+1} \oplus [\mathbf{c}[r+2, n]] \in \mathcal{K} \cap \mathcal{U}_{\mathbf{F}}(\mathbf{c}, r+1)$, we have $(0, \dots, 0) \in W_{\mathbf{c}, r+1}^{\mathbf{F}}(A_1, \dots, A_p)$. We are going to show that $(0, \dots, 0) \notin W_{\mathbf{c}, r}^{\mathbf{F}}(A_1, \dots, A_p)$. So, $W_{\mathbf{c}, r}^{\mathbf{F}}(A_1, \dots, A_p)$ is not convex.

Suppose $(0, \dots, 0) \in W_{\mathbf{c}, r}^{\mathbf{F}}(A_1, \dots, A_p)$. Then there exists $A = (a_{ij}) \in \mathcal{K} \cap \mathcal{U}_{\mathbf{F}}(\mathbf{c}, r)$. We have $a_{ii} = d$ for $1 \leq i \leq r+1$. By Lemma 2.6 b), we have

$$\sum_{i=1}^{r+1} c_i = \sum_{i=1}^{r+1} a_{ii} \leq \sum_{i=1}^{r+1} \lambda_i(A) \leq \sum_{i=1}^{r+1} c_i.$$

Hence, $\sum_{i=1}^{r+1} a_{ii} = \sum_{i=1}^{r+1} \lambda_i(A)$. By a result of Li [Li, Lemma 4.1], $a_{ij} = a_{ji} = 0$ for all $1 \leq i \leq r+1$ and $r+2 \leq j \leq n$. This implies that $\lambda_i(A) = d$ for $1 \leq i \leq r+1$. Since $A \in \mathcal{U}_{\mathbf{F}}(\mathbf{c}, r)$, we have $\sum_{i=1}^m d = \sum_{i=1}^m c_i$ for some $1 \leq m \leq r$. This gives

$$\begin{aligned} \sum_{i=1}^m \sum_{j=1}^{r+1} c_j &= \sum_{j=1}^{r+1} \sum_{i=1}^m c_i \\ \Rightarrow \sum_{i=1}^m \sum_{j=m+1}^{r+1} c_j &= \sum_{j=m+1}^{r+1} \sum_{i=1}^m c_i \\ \Rightarrow \sum_{i=1}^m \sum_{j=m+1}^{r+1} (c_i - c_j) &= 0 \\ \Rightarrow c_1 &= c_{r+1}, \end{aligned}$$

a contradiction. $\square$

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