A RELATION BETWEEN THE PLURICOMPLEX AND THE CLASSICAL GREEN FUNCTIONS IN THE UNIT BALL OF $\mathbf{C}^n$

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ABSTRACT. We give a sharp upper bound for the quotient of the pluricomplex and the classical Green functions in the unit ball of $\mathbf{C}^n$.

1. INTRODUCTION

The classical Green function for the unit ball $B = B(0, 1)$ in $\mathbf{C}^n$ or $\mathbf{R}^{2n}$ is

$$G_n(z, w) = \begin{cases} |w|^{2-2n} \left| z - w/|w|^2 \right|^{2-2n} - |z - w|^{2-2n} & \text{for } w \neq 0, \\ 1 - |z|^{2-2n} & \text{for } w = 0. \end{cases}$$

Here we defined it as a non-positive subharmonic function, whereas in most texts it is defined as the negative of our function (see for example [H], p. 77).

On the other hand, the pluricomplex Green function is $g_n(z, w) = \log |T_w(z)|$ where $T_w(z)$ is the Möbius transformation which maps w onto the origin. To be explicit,

$$T_w(z) = \frac{w - P_w(z) - (1 - |w|^2)^{1/2} Q_w(z)}{1 - \langle z, w \rangle}$$

where P_w is the orthogonal projection of $\mathbf{C}^n$ onto the subspace generated by w , and Q_w the projection onto the orthogonal complement of that subspace ([K], p. 148 and 224).

We define, for $n \geq 2$,

$$h_n(z, w) = \frac{g_n(z, w)}{G_n(z, w)}.$$

Note that h_n is a non-negative function, which can be extended to a continuous function on $B \times B$, since

$$\lim_{z, w \rightarrow \varsigma} h_n(z, w) = 0 \quad \text{for all } \varsigma \in B.$$

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2. THE RESULT

The following theorem was proved in [Ca].

Theorem 1. *For all $z, w \in B(0, 1) \subset \mathbf{C}^n$, $n \geq 2$, the inequality*

$$h_n(z, w) < 2^{2n-3}/(n-1)$$

holds. Moreover, this estimate is the best possible.

Proof. In the case $w = 0$, h_n reduces to $\log |z|/(1 - |z|^{2-2n})$. Let

$$f(x) = \frac{\log x}{1 - x^{2-2n}}$$

for all $x \in (0, 1)$. Then differentiation gives

$$f'(x) = \frac{x^{2n-2} + (2-2n)\log x - 1}{x^{2n-1}(1 - x^{2-2n})^2}.$$

By differentiating the numerator of $f'(x)$ we can easily see that it is decreasing and therefore greater than its value for $x = 1$, which is 0. Therefore $f(x)$ is increasing. Since $\lim_{x \rightarrow 1-} f(x) = 1/(2n-2) = 2^{-1}/(n-1) < 2^{2n-3}/(n-1)$, the theorem is proved in the case $w = 0$.

For the rest of the proof, assume $w \neq 0$. Let $\lambda = \langle z, w \rangle / |w|^2$. Then $P_w(z) = \lambda w$ and if we let $Q_w(z) = z^*$, then $z = \lambda w + z^*$. Hence

$$\begin{aligned} g_n(z, w) &= \frac{1}{2} \log |T_w(z)|^2 = \\ &= \frac{1}{2} \log \left| w - \lambda w - (1 - |w|^2)^{1/2} z^* \right|^2 - \frac{1}{2} \log |1 - \langle \lambda w + z^*, w \rangle|^2 = \\ &= \frac{1}{2} \log \left(|1 - \lambda|^2 |w|^2 + (1 - |w|^2) |z^*|^2 \right) - \frac{1}{2} \log |1 - \lambda|^2 |w|^2 \end{aligned}$$

using the orthogonality between z^* and w . By a similar argument we have

$$\begin{aligned} G_n(z, w) &= |w|^{2-2n} \left| \lambda w - w/|w|^2 + z^* \right|^{2-2n} - |(\lambda - 1)w + z^*|^{2-2n} = \\ &= |w|^{2-2n} \left(\left| \lambda w - w/|w|^2 \right|^2 + |z^*|^2 \right)^{1-n} - \left| |\lambda - 1|^2 |w|^2 + |z^*|^2 \right|^{1-n} = \\ &= \left(\left| \lambda w - w/|w|^2 \right|^2 |w|^2 + |z^*|^2 |w|^2 \right)^{1-n} - \left| |\lambda - 1|^2 |w|^2 + |z^*|^2 \right|^{1-n} = \\ &= \left(\left| \lambda |w|^2 - 1 \right|^2 + |z^*|^2 |w|^2 \right)^{1-n} - \left| |\lambda - 1|^2 |w|^2 + |z^*|^2 \right|^{1-n}. \end{aligned}$$

To simplify the notation, let $x = |w|^2$, $u = |z^*|^2 |w|^2$. After multiplying by $G_n < 0$ we can write the desired inequality in the following way

$$\begin{aligned} &\frac{1}{2} \log \left(|1 - \lambda|^2 x + (1 - x)u/x \right) - \frac{1}{2} \log |1 - \lambda x|^2 > \\ &> \frac{2^{2n-3}/(n-1)}{\left(|1 - \lambda x|^2 + u \right)^{n-1}} - \frac{2^{2n-3}/(n-1)}{\left(|1 - \lambda|^2 x + u/x \right)^{n-1}}. \end{aligned}$$

Since $1 > |z|^2 = |\lambda|^2 |w|^2 + |z^*|^2 = |\lambda|^2 x + u/x$, it suffices to show that this inequality holds for all $0 < x < 1$, $0 \leq u < x$ and $|\lambda| < \sqrt{x-u}/x$. For each $u \in [0, 1]$, define

$$f_{u,n}(t) = \log t + \frac{4^{n-1}/(n-1)}{(t+u)^{n-1}}.$$

Our inequality now takes the form

$$f_{u,n}(|1 - \lambda|^2 x + u/x - u) > f_{u,n}(|1 - \lambda x|^2).$$

Differentiation shows that $f_{u,n}(t)$ is decreasing when $(t+u)^n < 4^{n-1}t$. In the case $n = 2$, this occurs precisely when $t \in (2 - u - 2\sqrt{1-u}, 2 - u + 2\sqrt{1-u})$. We claim that when t belongs to this interval, then $f_{u,n}(t)$ is decreasing for all n . Indeed, as $(t+u)^2 < 4t$ and $0 < t < 4$, we have $(t+u)^n = [(t+u)^2]^{n/2} < (4t)^{n/2} \leq (4t)^{n/2}(4/t)^{n/2-1} = 4^{n-1}t$. Therefore it suffices to show that

$$2 - u - 2\sqrt{1-u} <_{(1)} |1 - \lambda|^2 x + u/x - u <_{(2)}$$

$$<_{(2)} |1 - \lambda x|^2 <_{(3)} 2 - u + 2\sqrt{1-u}$$

for all $0 < x < 1$, $0 \leq u < x$ and $|\lambda| < \sqrt{x-u}/x$. We prove the three inequalities separately.

$$(1) \quad 2 - u - 2\sqrt{1-u} < |1 - \lambda|^2 x + u/x - u$$

The proof is split into two cases.

Case 1. $\sqrt{x-u}/x \leq 1$.

Since $|1 - \lambda| \geq 1 - |\lambda|$, it is enough to prove (1) when λ is real and non-negative. Moreover, $0 \leq \lambda < \sqrt{x-u}/x$ for such λ , so it suffices to show (1) when $\lambda = \sqrt{x-u}/x$, i.e. to show that $(1 - \sqrt{x-u}/x)^2 x + u/x > 2 - 2\sqrt{1-u}$. Expanding the square and simplifying, we see that this is equivalent to $2\sqrt{1-u} > 1 + 2\sqrt{x-u} - x$. Both sides of this inequality are non-negative, so it is enough to prove the squared inequality, namely (after simplifying) $x + 3 > 4\sqrt{x-u}$. It is easy to check that this is true, even for $u = 0$.

Case 2. $\sqrt{x-u}/x > 1$, i.e. $u < x - x^2$.

In this case $|1 - \lambda|^2 x$ can be zero, so we have to show the inequality without that term, i.e. we must prove that $u/x > 2 - 2\sqrt{1-u}$, or equivalently $u/(2 - 2\sqrt{1-u}) > x$. If we view the left side of this inequality as a function of u , we can easily see, by differentiation, that it is decreasing and therefore greater than its value in $u = x - x^2$ which is $(x - x^2)/(2 - 2\sqrt{1-x+x^2})$. Elementary calculation shows that this expression is greater than or equal to x , and (1) is proved.

$$(2) \quad |1 - \lambda|^2 x + u/x - u < |1 - \lambda x|^2$$

By using the bound $|\lambda| < \sqrt{x-u}/x$ we obtain

$$\begin{aligned} |1 - \lambda|^2 x + \frac{u}{x} - u &< (1 + |\lambda|^2 - 2\operatorname{Re}(\lambda))x + \left(\frac{1}{x} - 1\right)(x - |\lambda|^2 x^2) = \\ &= 1 + |\lambda|^2 x^2 - 2\operatorname{Re}(\lambda)x = |1 - \lambda x|^2. \end{aligned}$$

$$(3) \quad |1 - \lambda x|^2 < 2 - u + 2\sqrt{1-u}$$

The same bound as used in the proof of (2) gives

$$|1 - \lambda x|^2 \leq (1 + |\lambda x|)^2 \leq (1 + \sqrt{x-u})^2 = 1 + x - u + 2\sqrt{x-u}.$$

Therefore

$$\begin{aligned} & |1 - \lambda x|^2 - (2 - u + 2\sqrt{1 - u}) \leq \\ & \leq 1 + x - u + 2\sqrt{x - u} - (2 - u + 2\sqrt{1 - u}) = \\ & = x - 1 + 2(\sqrt{x - u} - \sqrt{1 - u}) < 0 \end{aligned}$$

since $x < 1$. This completes the proof of (3).

It remains to show that the estimate is sharp. Let $w = (t, 0, \dots, 0)$ and $z = (-t, 0, \dots, 0)$, where t is real and positive. Then $\lambda = -1$ and $z^* = 0$. We substitute this in the expression for h_n . Then using Taylor's formula near $t = 1$ and the identity $a^m - b^m = (a - b)(a^{m-1} + a^{m-2}b + \dots + b^{m-1})$ we get

$$\begin{aligned} h_n(z, w) &= \frac{\log(2t) - \log(1 + t^2)}{(1 + t^2)^{2-2n} - (2t)^{2-2n}} = \\ &= - \frac{(2t)^{2n-2}(1 + t^2)^{2n-2}}{((2t)^2)^{n-2} + ((2t)^2)^{n-3}(1 + t^2)^2 + \dots + ((1 + t^2)^2)^{n-2}} \times \\ &\times \frac{\log 2 + (t - 1) - (t - 1)^2/2 + O(t - 1)^3 - (\log 2 + (t - 1) + O(t - 1)^3)}{(1 - t)^2(1 + t)^2} = \\ &= - \frac{(2t)^{2n-2}(1 + t^2)^{2n-2}}{((2t)^2)^{n-2} + ((2t)^2)^{n-3}(1 + t^2)^2 + \dots + ((1 + t^2)^2)^{n-2}} \times \\ &\times \left[-\frac{1}{2} + O(t - 1) \right] \frac{1}{(1 + t)^2} \rightarrow \\ &\rightarrow - \frac{2^{2n-2} \cdot 2^{2n-2}}{2^2 \left[(2^2)^{n-2} (n - 1) \right]} \cdot \left(-\frac{1}{2} \right) = \frac{2^{2n-3}}{n - 1} \end{aligned}$$

when $t \rightarrow 1$. □

Remark. The example at the end of the proof shows that $h_n(z, w)$ tends to its supremum, when z and w approach opposite points at the boundary. Since also $h_n(z, w)$ has its least value 0 for $z = w$, we conjecture that

$$\sup_{|z| \leq |w|} h_n(z, w) = h_n(-w, w)$$

for all $w \in B$.

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