PROCEEDINGS OF THE
AMERICAN MATHEMATICAL SOCIETY
Volume 125, Number 6, June 1997, Pages 1779–1783
S 0002-9939(97)03903-8

A NEW FIXED POINT THEOREM AND ITS APPLICATIONS

XIAN WU

(Communicated by Palle E. T. Jorgensen)

Abstract. In this paper, we first give a new fixed theorem of lower semicontinuous multivalued mappings, and then, as its applications we obtain some new equilibrium theorems for abstract economies and qualitative games.

1. Introduction and preliminaries

In 1972, Himmelberg [4] proved a well-known fixed point theorem as follows:

Theorem A. *Let X be a convex subset of a locally convex Hausdorff topological vector space and D be a nonempty compact subset of X. Let $T : X \to 2^D$ be an upper semicontinuous correspondence such that for each $x \in X$, $T(x)$ is a nonempty closed convex subset of D. Then there exists a point $\overline{x} \in D$ such that $\overline{x} \in T(\overline{x})$.*

Theorem A is called a Himmelberg fixed point theorem. It generalizes the well-known Kakutani-Fan-Glicksberg fixed point theorem.

In recent years, many mathematicians have studied fixed point problems of lower semicontinuous multivalued mappings. But, up to the present, there is no ideal result (for example, see also [1, 8]) and P. Cubioth [1] pointed out that the fixed point theorem for lower semicontinuous multivalued mappings in [8] is not right.

In this paper, we first give a new fixed point theorem of lower semicontinuous multivalued mappings. It can almost compare with Himmelberg's fixed point theorem. Next, as its applications we give some new equilibrium existence theorems for abstract economies and qualitative games. These equilibrium existence theorems differ from those which have appeared in other papers (for example, see also [9] and the references therein).

In the following, we give some basic concepts and notations.

Let A be a subset of a topological space X. We shall denote by 2^A the family of all subsets of A and by $\overline{A}$ the closure of A in X. If A is a subset of a topological vector space X, we shall denote by $\overline{\text{co}}A$ the closed convex hull of A.

Let X, Y be two topological spaces and $T : X \to 2^Y$ be a multivalued mapping. T is said to be upper semicontinuous (respectively, almost upper semicontinuous)

Received by the editors December 27, 1995.
1991 *Mathematics Subject Classification.* Primary 47H04, 47H10, 52A07, 90D13.
Key words and phrases. Locally convex topological vector space, metrizable, lower semicontinuous, qualitative game, abstract economy.
The author was supported by the Foundation of Yunnan Sci. Tech. Commission and the Foundation of Yunnan Educational Commission.

if for each $x \in X$ and each open set V in Y with $T(x) \subset V$, there exists an open neighborhood U of x in X such that $T(z) \subset V$ (respectively, $T(z) \subset \overline{V}$) for each $z \in U$. T is said to be lower semicontinuous if for each open set V in Y, the set $\{x \in X : T(x) \cap V \neq \varnothing\}$ is open in X. Obviously, T is lower semicontinuous if and only if for each closed set M in Y, the set $\{x \in X : T(x) \subset M\}$ is closed in X. T is said to have open upper sections if $T(x)$ is open in Y for each $x \in X$.

Let I be a set of agents. For each $i \in I$, let X_i be a nonempty set of actions. Let $X = \prod_{i \in I} X_i$. An abstract economy (or a generalized game) $\Gamma = (X_i, A_i, B_i, P_i)_{i \in I}$ is defined as a family of ordered quadruples (X_i, A_i, B_i, P_i), where X_i is a topological space (a choice set), $A_i, B_i : X \to 2^{X_i}$ are constraint correspondences and $P_i : X \to 2^{X_i}$ is a preference correspondence. An equilibrium of Γ is a point $\overline{x} \in X$ such that $\overline{x}_i \in \overline{B_i(\overline{x})}$ and $A_i(\overline{x}) \cap P_i(\overline{x}) = \varnothing$. $\Gamma = (X_i, P_i)_{i \in I}$ is a qualitative game if for each player $i \in I$, X_i is a strategy set of player i, and $P_i : X \to 2^{X_i}$ is a preference correspondence of player i. A maximal element of Γ is a point $\overline{x} \in X$ such that $P_i(\overline{x}) = \varnothing$ for each $i \in I$.

2. A fixed point theorem

Theorem 1. *Let I be an index set. For each $i \in I$, let X_i be a nonempty convex subset of a Hausdorff locally convex topological vector space E_i, D_i a nonempty compact metrizable subset of X_i and S_i, $T_i : X := \prod_{i \in I} X_i \to 2^{D_i}$ two multivalued mappings with the following conditions:*

(i) *for each $x \in X$, $\overline{\mathrm{co}} S_i(x) \subset T_i(x)$ and $S_i(x) \neq \varnothing$,*

(ii) *S_i is lower semicontinuous.*

Then there exists a point $\overline{x} = \prod_{i \in I} \overline{x}_i \in D := \prod_{i \in I} D_i$ such that $\overline{x}_i \in T_i(\overline{x})$ for each $i \in I$.

Proof. Since D_i is compact, $D = \prod_{i \in I} D_i$ is also compact in X, and hence $\mathrm{co}\, D$ is paracompact in X by Lemma 1 in [2]. For each $i \in I$, since again $S_i : X \to 2^{D_i}$ is lower semicontinuous and $S_i(x) \neq \varnothing$ for each $x \in X$, the mapping $\overline{\mathrm{co}} S_i : X \to 2^{D_i}$ defined by $\overline{\mathrm{co}} S_i(x) = \overline{\mathrm{co}}[S_i(x)]$ for each $x \in X$, is lower semicontinuous by Proposition 2.3 and Proposition 2.6 in [6] and obviously, every $\overline{\mathrm{co}} S_i(x)$ is nonempty and complete. By Theorem 1.1 in Michael [5], there exists an upper semicontinuous multivalued mapping $H_i : \mathrm{co}\, D \to 2^{D_i}$ with nonempty values such that $H_i(x) \subset \overline{\mathrm{co}} S_i(x)$ for all $x \in \mathrm{co}\, D$.

For each $x \in \mathrm{co}\, D$, let $P_i(x) = \overline{\mathrm{co}} H_i(x)$. Then $P_i : \mathrm{co}\, D \to 2^{D_i}$ is an upper semicontinuous multivalued mapping with nonempty closed convex values by Lemma 1 and Lemma 2 in [7] and $P_i(x) \subset \overline{\mathrm{co}} S_i(x) \subset T_i(x)$ for all $x \in \mathrm{co}\, D$. Define a multivalued mapping $P : \mathrm{co}\, D \to 2^D$ by

$$P(x) = \prod_{i \in I} P_i(x), \quad \forall x \in \mathrm{co}\, D.$$

Then $P : \mathrm{co}\, D \to 2^D$ is an upper semicontinuous multivalued mapping with nonempty closed convex values by Lemma 3 in [3]. Therefore, by Himmelberg's fixed point theorem [4], there exists $\overline{x} = \prod_{i \in I} \overline{x}_i \in D$ such that $\overline{x} \in P(\overline{x})$, i.e. $\overline{x}_i \in P_i(\overline{x})$ for all $i \in I$. Hence $\overline{x}_i \in T_i(\overline{x})$ for all $i \in I$. This completes the proof.

Corollary 2. *Let X be a nonempty convex subset of a Hausdorff locally convex topological vector space E, D a nonempty compact metrizable subset of X and S, $T : X \to 2^D$ two multivalued mappings with the following conditions*:

(i) *for each $x \in X$, $\overline{\mathrm{co}} S(x) \subset T(x)$ and $S(x) \neq \varnothing$,*

(ii) *S is lower semicontinuous.*
Then there exists a point $\overline{x} \in D$ such that $\overline{x} \in T(\overline{x})$.

Corollary 3. *Let X be a nonempty convex subset of a Hausdorff locally convex topological vector space E, D a nonempty compact metrizable subset of X and $T : X \to 2^D$ a multivalued mapping with the following conditions*:
(i) *for each $x \in X$, $T(x)$ is a nonempty closed convex set,*
(ii) *T is lower semicontinuous.*
Then there exists a point $\overline{x} \in D$ such that $\overline{x} \in T(\overline{x})$.

Remark. Corollary 3 can almost compare with Himmelberg's fixed point theorem.

3. Equilibrium existence theorems for abstract economies

In this section, we give some new equilibrium existence theorems for abstract economies.

Theorem 4. *Let $\Gamma = (X_i, A_i, B_i, P_i)_{i\in I}$ be an abstract economy such that for each $i \in I$, the following conditions are fulfilled*:
(i) *X_i is a nonempty convex subset of a Hausdorff locally convex topological vector space E_i and D_i is a nonempty compact metrizable subset of X_i,*
(ii) *for each $x \in X := \prod_{i\in I} X_i, P_i(x) \subset D_i, A_i(x) \subset B_i(x) \subset D_i$ and $B_i(x)$ is nonempty convex,*
(iii) *the set $W_i := \{x \in X : A_i(x) \cap P_i(x) \neq \varnothing\}$ is closed in X,*
(iv) *the mappings $A_i|_{W_i}, P_i|_{W_i} : W_i \to 2^{D_i}$ are lower semicontinuous and either A_i or $P_i : X \to 2^{D_i}$ has open upper sections, $B_i : X \to 2^{D_i}$ is lower semicontinuous,*
(v) *for each $x \in X, x_i \notin \overline{\mathrm{co}}(A_i(x) \cap P_i(x))$.*
Then there exists a point $\overline{x} \in D := \prod_{i\in I} D_i$ such that $\overline{x}_i \in \overline{B_i(\overline{x})}$ and $A_i(\overline{x}) \cap P_i(\overline{x}) = \varnothing$ for all $i \in I$, i.e. $\overline{x}$ is an equilibrium point of Γ.

Proof. For each $x \in X$, let

$$S_i(x) = \begin{cases} A_i(x) \cap P_i(x), & \text{if } x \in W_i, \\ B_i(x), & \text{if } x \notin W_i, \end{cases}$$

$$T_i(x) = \begin{cases} \overline{\mathrm{co}}(A_i(x) \cap P_i(x)), & \text{if } x \in W_i, \\ \overline{B_i(x)}, & \text{if } x \notin W_i, \end{cases}$$

$i \in I$. Then $S_i, T_i : X \to 2^{D_i}$ are two multivalued mappings with nonempty values and $\overline{\mathrm{co}} S_i(x) \subset T_i(x)$ for all $x \in X$. In the following, we shall prove that $S_i : X \to 2^{D_i}$ is lower semicontinuous.

For each closed set V in D_i, the set

$$\begin{aligned} \{x \in X : S_i(x) \subset V\} &= \{x \in W_i : A_i(x) \cap P_i(x) \subset V\} \cup \{x \in X \setminus W_i : B_i(x) \subset V\} \\ &= \{x \in W_i : A_i(x) \cap P_i(x) \subset V\} \cup \{x \in X : B_i(x) \subset V\}. \end{aligned}$$

By Fact 6.1 in [10] we know that the mapping $(A_i \cap P_i)|_{W_i} : W_i \to 2^{D_i}$ defined by

$$(A_i \cap P_i)|_{W_i}(x) = A_i(x) \cap P_i(x), \quad \forall x \in W_i,$$

is lower semicontinuous. Consequently, the set $\{x \in W_i : A_i(x) \cap P_i(x) \subset V\}$ is closed in W_i, and hence it is also closed in X because W_i is closed in X. Since again $B_i : X \to 2^{D_i}$ is lower semicontinuous, the set $\{x \in X : B_i(x) \subset V\}$ is also closed in X. Therefore, the set $\{x \in X : S_i(x) \subset V\}$ is closed in X. It shows that $S_i : X \to 2^{D_i}$ is lower semicontinuous.

By virtue of Theorem 1, there exists a point $\overline{x} = \prod_{i \in I} \overline{x}_i \in D = \prod_{i \in I} D_i$ such that

$$\overline{x}_i \in T_i(x), \quad \forall i \in I.$$

By condition (v) we know that

$$\overline{x}_i \in \overline{B_i(\overline{x})} \quad \text{and} \quad A_i(\overline{x}) \cap P_i(\overline{x}) = \varnothing$$

for all $i \in I$, i.e. $\overline{x} \in D$ is an equilibrium point of Γ.

Theorem 5. *Let $\Gamma = (X_i, A_i, B_i, P_i)_{i \in I}$ be an abstract economy such that for each $i \in I$, the following conditions are fulfilled*:

(i) *X_i is a nonempty convex subset of a Hausdorff locally convex topological vector space E_i and D_i is a nonempty compact metrizable subset of X_i,*

(ii) *for each $x \in X := \prod_{i \in I} X_i, P_i(x) \subset D_i, A_i(x) \subset B_i(x) \subset D_i$ and $B_i(x)$ is nonempty convex,*

(iii) *the mapping $H_i : X \to 2^{D_i}$ defined by*

$$H_i(x) = A_i(x) \cap P_i(x), \quad \forall x \in X,$$

is lower semicontinuous,

(iv) *the mapping $B_i : X \to 2^{D_i}$ is lower semicontinuous,*

(v) *for each $x \in X, x_i \notin \overline{\mathrm{co}}(A_i(x) \cap P_i(x))$.*

Then there exists a point $\overline{x} \in D := \prod_{i \in I} D_i$ such that $\overline{x}_i \in \overline{B_i(\overline{x})}$ and $A_i(\overline{x}) \cap P_i(\overline{x}) = \varnothing$ for all $i \in I$, i.e. $\overline{x}$ is an equilibrium point of Γ.

Proof. For each $i \in I$, let

$$W_i = \{x \in X : A_i(x) \cap P_i(x) \neq \varnothing\}.$$

Then by condition (iii) we know that W_i is open in X. For each $x \in X$, let

$$S_i(x) = \begin{cases} A_i(x) \cap P_i(x), & \text{if } x \in W_i, \\ B_i(x), & \text{if } x \notin W_i, \end{cases}$$

$$T_i(x) = \begin{cases} \overline{\mathrm{co}}(A_i(x) \cap P_i(x)), & \text{if } x \in W_i, \\ \overline{B_i(x)}, & \text{if } x \notin W_i, \end{cases}$$

$i \in I$. Then $S_i, T_i : X \to 2^{D_i}$ are two multivalued mappings with nonempty values and $\overline{\mathrm{co}} S_i(x) \subset T_i(x)$ for all $x \in X$. In the following, we shall prove that $S_i : X \to 2^{D_i}$ is lower semicontinuous.

For each closed set V in D_i, the set

$$\begin{aligned} \{x \in X : S_i(x) \subset V\} &= \{x \in W_i : A_i(x) \cap P_i(x) \subset V\} \cup \{x \in X \setminus W_i : B_i(x) \subset V\} \\ &= \{x \in X : A_i(x) \cap P_i(x) \subset V\} \cup \{x \in X \setminus W_i : B_i(x) \subset V\}. \end{aligned}$$

By condition (iii), the set $\{x \in X : A_i(x) \cap P_i(x) \subset V\}$ is closed in X. Since again W_i is open in X, $X \setminus W_i$ is closed in X, and hence the set $\{x \in X \setminus W_i : B_i(x) \subset V\}$ is closed in X because $B_i : X \to 2^{D_i}$ is lower semicontinuous. Therefore, the set $\{x \in X : S_i(x) \subset V\}$ is closed in X. It shows that $S_i : X \to 2^{D_i}$ is lower semicontinuous.

By virtue of Theorem 1, there exists a point $\overline{x} = \prod_{i \in I} \overline{x}_i \in D = \prod_{i \in I} D_i$ such that

$$\overline{x}_i \in T_i(\overline{x}), \quad \forall i \in I.$$

By condition (v) we know that

$$\overline{x}_i \in \overline{B_i(\overline{x})} \quad \text{and} \quad A_i(\overline{x}) \cap P_i(\overline{x}) = \varnothing$$

for all $i \in I$, i.e. $\overline{x} \in D$ is an equilibrium point of Γ.

In Theorem 4 and Theorem 5, when $A_i(x) = B_i(x) = X_i$ for each $x \in X$ and $i \in I$, we can obtain the following Theorem 6 and Theorem 7, respectively.

Theorem 6. *Let $\Gamma = \{X_i, P_i : i \in I\}$ be a qualitative game such that for each $i \in I$, the following conditions hold:*

(i) *X_i is a nonempty compact metrizable convex subset of a Hausdorff locally convex topological vector space E_i,*

(ii) *the set $W_i := \{x \in X : P_i(x) \neq \varnothing\}$ is closed in X,*

(iii) *$P_i|_{W_i} : W_i \to 2^{X_i}$ is lower semicontinuous,*

(iv) *for each $x \in X$, $x_i \notin \overline{\text{co}} P_i(x)$.*

Then there is a maximal element of the game Γ; i.e. there exists a point $\overline{x} \in X$ such that $P_i(\overline{x}) = \varnothing$ for all $i \in I$.

Theorem 7. *Let $\Gamma = \{X_i, P_i : i \in I\}$ be a qualitative game such that for each $i \in I$, the following conditions hold*:

(i) *X_i is a nonempty compact metrizable convex subset of a Hausdorff locally convex topological vector space E_i,*

(ii) *$P_i : X \to 2^{X_i}$ is lower semicontinuous,*

(iii) *for each $x \in X$, $x_i \notin \overline{\text{co}} P_i(x)$.*

Then there is a maximal element of the game Γ.

An open problem

Is the conclusion of Corollary 3 tenable if the metrizable condition of D are canceled?

References

1. P. Cubiotti, *Some remarks on fixed points of lower semicontinuous multifunctions*, J. Math. Anal. Appl. **174** (1993), 407–412. MR **94c:**47081
2. X. P. Ding, W. K. Kim and K. K. Tan, *A selection theorem and its applications*, Bull. Austral. Math. Soc.**46** (1992), 205–212. MR **93g:**47072
3. K. Fan, *Fixed point and minimax theorems in locally convex topological linear spaces*, Proc. Nat. Acad. Sci. U.S.A. **38** (1952), 121-126. MR **13:**858d
4. C. J. Himmelberg, *Fixed points of compact multifunctions*, J. Math. Anal. Appl. **38** (1972), 205–207. MR **46:**2505
5. E. Michael, *A theorem on semi-continuous set-valued functions*, Duke Math. J. **26** (1959), 647–651. MR **22:**229
6. E. Michael, *Continuous selections*, I, Ann. of Math. (2) **63** (1956), 361–382. MR **17:**990e
7. D. I. Rim and W. K. Kim, *A fixed point theorem and existence of equilibrium for abstract economies*, Bull. Austral. Math. Soc. **45** (1992), 385-394. MR **93g:**47074
8. G. Tian, *Fixed point theorems for mappings with noncompact and nonconvex domains*, J. Math. Anal. Appl. **158** (1991), 161–167. MR **92e:**47110
9. K. K. Tan and X. Z. Yuan, *Existence of equilibrium for abstract economies*, J. Math. Econom. **23** (1994), 243–251. MR **95h:**90024
10. N. C. Yannelis and N. D. Prabhakar, *Existence of maximal elements and equilibra in linear topological spaces*, J. Math. Econom. **12** (1983), 223–245; **13** (1984), 305. MR **87h:**90061a,b

Department of Mathematics, Yunnan Normal University, Kunming, Yunnan 650092, People's Republic of China