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WHEN IS A p-BLOCK A q-BLOCK?

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Abstract. Let p and q be distinct prime numbers and let G be a finite group. If B_p is a p-block of G and B_q is a q-block, we study when the set of ordinary irreducible characters in the blocks B_p and B_q coincide.

1. Introduction

Let p and q be distinct prime numbers and let G be a finite group. Since the representation theory of G in every prime characteristic is connected to that in characteristic zero via the decomposition map, there is a relationship between the theories in characteristics p and q. However, one hardly finds anything in the literature devoted to this connection.

In this note, we investigate the situation when the set $\mathrm{Irr}(B_p)$ of irreducible ordinary characters of a p-block B_p of G coincides with $\mathrm{Irr}(B_q)$, where B_q is a q-block. This happens, for instance, if the group G has an irreducible character χ which is of p- and q-defect zero, that is, with $|G|_p|G|_q$ dividing $\chi(1)$. This is equivalent to the fact that $\mathrm{Irr}(B_p) = \mathrm{Irr}(B_q)$ consists only of one character χ.

Here we shall give some evidence that this is the only possibility for p- and q-blocks to coincide. We state the

Conjecture. *If* $\mathrm{Irr}(B_p) = \mathrm{Irr}(B_q)$, *then* $|\mathrm{Irr}(B_p)| = 1$.

In what follows, we give an affirmative answer in the case that G is "solvable for one prime" or that one of the blocks contains only one Brauer character.

We would like to remark that one should be able to prove the conjecture for principal blocks. In this case, the problem reduces to simple groups and then a matter of checking will give the answer. We have not attempted to do this, however.

Finally, we are indebted to M. Isaacs for the shortened proof we present here.

2. Proofs

Our objective is to prove the following result.

(2.1) Theorem. *Let p and q be distinct primes and let G be a finite group. Let B_p be a p-block and let B_q be a q-block of G with* $\mathrm{Irr}(B_p) = \mathrm{Irr}(B_q)$. *If G is p-solvable or q-solvable, then* $|\mathrm{Irr}(B_p)| = 1$.

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We need a result of independent interest. As a consequence of it, notice that the characters in a block of positive defect cannot all be conjugate under the action of $\mathrm{Aut}(G)$.

(2.2) Proposition. *Let N be a normal subgroup of G and let b be a p-block of N. If $\mathrm{Irr}(b)$ consists of G-conjugates of some $\theta \in \mathrm{Irr}(N)$, then $|\mathrm{Irr}(b)| = 1$.*

Proof. By replacing θ by some conjugate, certainly we may assume that $\theta \in \mathrm{Irr}(b)$. Write $|\mathrm{Irr}(b)| = k$ and first of all, observe that $\theta^g \in \mathrm{Irr}(b)$ if and only if g is an element of the stabilizer $I(b)$ of b in G. Therefore,

$$k = |\mathrm{Irr}(b)| = |I(b) : I(\theta)|,$$

where $I(\theta)$ denotes the stabilizer of θ in G. Hence, notice that we may assume that $I(b) = G$.

Now, by the Weak Block Orthogonality (3.6.21 of [2]), we have that

$$\sum_{\alpha \in \mathrm{Irr}(b)} \alpha(1)\alpha(n) = 0$$

for all p-elements $1 \neq n \in N$. Since all characters in b have the same degree $\theta(1)$, we conclude that

$$\sum_{\alpha \in \mathrm{Irr}(b)} \alpha$$

is a character of N which vanishes on its p-elements, so we conclude that $|N|_p$ divides its degree $k\theta(1)$. Now, by Theorem (5.5.17) of [2], we know that there is a character $\xi \in \mathrm{Irr}(b)$ such that $|G : I(\xi)|$ is not divisible by p. By the hypothesis, ξ is some conjugate of θ and we conclude that $k = |G : I(\theta)|$ is a p'-number. Therefore, $|N|_p$ divides $\theta(1)$, as required. □

We will derive Theorem (2.1) from the following.

(2.3) Theorem. *Let G be a finite group and let π be a set of primes such that for every chief factor of G there exists a prime in π not dividing the order of the factor. Let $\mathcal{X} \subseteq \mathrm{Irr}(G)$ be such that for every $p \in \pi$, there is a p-block B_p of G satisfying $\mathrm{Irr}(B_p) = \mathcal{X}$. Then $|\mathcal{X}| = 1$.*

Proof. If $N \triangleleft G$ and $\mathcal{Y} \subseteq \mathrm{Irr}(N)$ is the set of irreducible constituents of χ_N, where χ runs through $\mathcal{X}$, we prove by induction on $|N|$ that every member of $\mathcal{Y}$ has p-defect zero for all $p \in \pi$. Notice that if this is true, the theorem follows by putting $N = G$.

Certainly, we may assume that $N > 1$. We fix $p \in \pi$ and $\theta \in \mathcal{Y}$ and we prove that θ has p-defect zero. Let N/M be a chief factor of G. By the inductive hypothesis, we have that if $\mathcal{Z} \subseteq \mathrm{Irr}(M)$ is the set of irreducible constituents of χ_M, where χ runs through $\mathcal{X}$, every member of $\mathcal{Z}$ has defect zero for every prime in π.

Now, let $\eta \in \mathrm{Irr}(M)$ under θ and notice that η has p-defect zero, since $\eta \in \mathcal{Z}$. Therefore, we may assume that N/M is divisible by p because otherwise

$$\theta(1)_p = \eta(1)_p = |M|_p = |N|_p$$

and this would prove the theorem. By hypothesis, choose $q \in \pi$ to be a prime not dividing $|N/M|$ and let b_q be the q-block of θ. Since η has q-defect zero, then θ has q-defect zero and then $b_q = \{\theta\}$.

Now, let b_p be the p-block of θ and notice that, by Proposition (2.2), it is enough to show that b_p consists of G-conjugates of θ.

Let $\chi \in \mathcal{X}$ lying over θ and let B_p and B_q be the p- and q-block of χ, respectively. Therefore, B_p covers b_p, B_q covers b_q and $\mathrm{Irr}(B_p) = \mathrm{Irr}(B_q)$. Let $\gamma \in \mathrm{Irr}(b_p)$. Since B_p covers the block b_p, by (5.5.8.ii) of [2], we may find $\xi \in \mathrm{Irr}(B_p)$ lying over γ. Now, $\xi \in \mathrm{Irr}(B_q)$ and then the q-block of γ is covered by B_q. But, since B_q covers $b_q = \{\theta\}$, the q-block of γ is just $\{\gamma\}$ and by (5.5.3) of [2], γ and θ are G-conjugate. This completes the proof of the theorem. □

Proof of Theorem (2.1). If $\mathcal{X} = \mathrm{Irr}(B_p) = \mathrm{Irr}(B_q)$, the hypotheses of Theorem (2.3) are satisfied with $\pi = \{p, q\}$. □

(2.4) Theorem. *Let p and q be distinct primes and let B_p and B_q be a p-block and a q-block of G, respectively. Suppose again that* $\mathrm{Irr}(B_p) = \mathrm{Irr}(B_q)$. *If one of the blocks contains only one Brauer character, then* $|\mathrm{Irr}(B_p)| = 1$.

Proof. Assume, for instance, that $|\mathrm{IBr}(B_p)| = 1$ and consider the character

$$\rho = \sum_{\chi \in \mathrm{Irr}(B_p)} \chi(1)\chi = \sum_{\chi \in \mathrm{Irr}(B_q)} \chi(1)\chi\,.$$

If ρ^0 denotes the restriction of ρ to the to p-regular elements of G and ϕ is the unique Brauer character of B_p, then

$$\rho^0 = e\phi$$

for some integer e.

As ρ is the regular character of the q-block B_q, we have in particular that $\rho^0(x) = 0$ for all q-elements $x \in G$. Thus $\phi(x) = 0$ for all q-elements $1 \neq x \in G$, and consequently, $\chi(x) = 0$ for all q-elements $1 \neq x \in G$ and $\chi \in \mathrm{Irr}(B_p) = \mathrm{Irr}(B_q)$. This means that χ has q-defect zero ([2], 3.6.27), and we are done. □

Remark. To attack the conjecture for arbitrary groups and arbitrary blocks, Brauer's dimensional formula (see [1])

$$\dim(B_p) = p^{2a-d}(u_{B_p})^2 v_{B_p},$$

where $|G|_p = p^a$, d is the defect of B_p, $u_{B_p} = \gcd\{\chi(1)_{p'} \mid \chi \in \mathrm{Irr}(B_p)\}$ and p does not divide v_{B_p}, might be the key.

If $\mathrm{Irr}(B_p) = \mathrm{Irr}(B_q)$, one has to show that q does not divide the mysterious invariant v_{B_p}. This seems to be difficult, however. Actually, if B_p contains only one Brauer character, then $v_{B_p} = 1$ (see [1]) and this gives another proof of (2.4).

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