$\mathcal{L}$ -CLASSES ON PSEUDOMANIFOLDS WITH ONE SINGULAR STRATUM

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ABSTRACT. We study the index theorem and Chern character of an admissible pseudomanifold $X^\dagger$ with one singular stratum. Under a condition on the link, we give a de Rham type realization of the Goresky-MacPherson-Siegel $\mathcal{L}$ -classes on $X^\dagger$ in terms of curvature forms and eta invariant of the link.

0. INTRODUCTION

In [CST] and [MW1] Connes-Sullivan-Teleman and Moscovici-Wu solved a long-standing problem of recovering the topological Pontryagin classes from local data. In [MW1] and [MW2], using the finite propagation speed property in the unbounded picture, Moscovici-Wu gave a realization of the $\mathcal{L}$ -classes, hence of the Pontryagin classes, of the topological manifold in terms of Alexander-Spanier cycles by looking at the “straight” Chern character of [CM].

In this paper, we will generalize the result in [MW1] to a particular type of pseudomanifolds, namely $X^\dagger = M \cup (c^\dagger(L) \times N)$. As in [C1] and [GM], for spaces with singularities, we work with “characteristic classes” in homology rather than cohomology. In [GM] Goresky and MacPherson defined $\mathcal{L}$ -classes for pseudomanifolds with even codimensional strata. By the work of Siegel [Si], one can extend this definition to Witt spaces. Our approach differs from [MW1] in the sense that we relate the “straight” Chern character of D to $\mathcal{L}$ -class directly. This asserts that (Theorem 3.4) the Goresky-MacPherson-Siegel $\mathcal{L}$ -class of $X^\dagger$ (with $2L$ having zero oriented cobordism) is represented by the following cycle:

$$L_*(X^\dagger; \rho_{X^\dagger}) = 2^{2m'} \mathcal{L}_{m'}(R(g^M)) \oplus -\delta_\ell \eta(L) 2^{2n'} \mathcal{L}_{n'}(R(g^N)),$$

where $m' = \frac{m-*}{4}$, $n' = \frac{n-*}{4}$ and $\delta_\ell = \frac{1}{2}(1 - (-1)^\ell)$. Furthermore, we will show that the Chern character of D coincides with the “straight” Chern character of D .

In [BC] and [C2], Bismut and Cheeger defined $\mathcal{L}$ -classes by means of the index pairing. Theorem 4.1 below shows that the Bismut-Cheeger $\mathcal{L}$ -classes and Goresky-MacPherson-Siegel $\mathcal{L}$ -classes are the same (up to constants) for spaces $X^\dagger$ defined above.

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1. PRELIMINARIES

1.1. Pseudomanifolds. For your reference, let's recall some definitions in [C1]. An m -dimensional pseudomanifold Y^m is a finite simplicial complex such that every point is contained in a closed m -simplex and every $(m-1)$ -simplex is a face of either one or two m -simplices. We also endow Y^m with a metric, which determines a distance function ρ_Y , and assume that $Y \setminus \sum^{m-2}$ is a flat manifold (in the induced metric) where $\sum^{m-2}$ is the $(m-2)$ -skeleton associated with a triangulation. A pseudomanifold is called admissible [C1, p.127] if the middle L^2 -cohomology group of even dimensional links, in the Riemannian handle decomposition, vanishes.

After the construction of Fredholm module in Section 2, we will consider a particular type of pseudomanifolds:

Let M be a smooth, oriented, compact and connected m -dimensional manifold with boundary $\partial M = L \times N$ where L and N are smooth, oriented and compact manifolds (without boundary) of dimensions ℓ and n respectively, and either ℓ is odd or $H^{\frac{\ell}{2}}(L) = 0$.

In this paper, we will use the following notation:

$$\begin{aligned} c_{0,\infty}(L) &= (0, \infty) \times L && \text{infinite cone with link } L, \\ c(L) &= (0, 1) \times L && \text{cone with link } L, \\ \dagger &= \text{the tip of the cone,} \\ c^\dagger(L) &= c(L) \cup \{\dagger\} && \text{completed cone with link } L, \\ X &= M \cup (c(L) \times N) && \text{regular part,} \\ X^\dagger &= M \cup (c^\dagger(L) \times N) && \text{pseudomanifold with one singular stratum.} \end{aligned}$$

We will endow $X^\dagger$ with a metric g (not necessary flat on X) such that:

- (i) g is a measurable metric on $X^\dagger$;
- (ii) $g|_M$ is a smooth metric on M and is a product near ∂M ;
- (iii) $g|_{c^\dagger(L) \times N} = (dr^2 + \psi(r)^2 g^L) \oplus g^N$, where g^L and g^N are smooth metrics on L and N respectively, and $\psi : [0, 1] \rightarrow [0, 1]$ is a C^∞ function such that

$$\psi(r) = \begin{cases} r, & r \in [0, \frac{2}{3}], \\ 1, & r \in [\frac{3}{4}, 1], \end{cases}$$

and $\psi(r) \neq 0$ for $r > 0$.

Clearly, this is a Lipschitz metric.

In order not to change the metric (i.e. $\rho_{X^\dagger}|_{X \times X} = \rho_X$), L must be connected. Otherwise, we will need to add a cone on each component of L . In other words, we have to consider the normalization of the space [GM, p.151]. So in the rest of this paper, we will assume that L and N are connected. In this case, $X^\dagger$ is the metric completion of X .

1.2. Sullivan complex. In the presence of singularities it is more convenient to use the Sullivan complex ([BC], [MW2] and [Su]). Let's recall the corresponding results [MW2] for the space $X^\dagger$. Let $pr : c(L) \times N \rightarrow N$ be the projection onto the second factor and $j : L \times N \rightarrow M$ be the inclusion map. The complex of stratified differential forms on $X^\dagger$ is:

$$\Omega^*(X^\dagger)_{SA} = \{\omega \in \Omega^*(X) : \omega|_{c(L) \times N} = pr^*(\tilde{\omega}), \quad \tilde{\omega} \in \Omega^*(N)\}.$$

$\Omega^0(X^\dagger)_{SA}$ is an algebra, which will also be denoted by $C_{SA}^\infty(X^\dagger)$. The differentials are the usual ones. From [MW2], $H^*(X^\dagger) \cong H_*(\Omega^*(X^\dagger)_{SA}, d)$. For homology, there is also a chain complex of de Rham type, namely

$$\Omega_q(X^\dagger)_{SA} = \Omega^{m-q}(M) \oplus \Omega^{n-q}(N)$$

with the boundary operator $\partial : \Omega_q(X^\dagger)_{SA} \rightarrow \Omega_{q-1}(X^\dagger)_{SA}$ given by

$$\partial(\omega_1, \omega_2) = \left((-1)^q d\omega_1, (-1)^q d\omega_2 + \int_L j^* \omega_1 \right).$$

Again from [MW2], $H_*(X^\dagger) \cong H_*(\Omega_*(X^\dagger)_{SA}, \partial)$.

For these complexes, there is a pairing

$$\begin{aligned} \Omega_q(X^\dagger)_{SA} \otimes \Omega^q(X^\dagger)_{SA} &\longrightarrow \mathbb{C}, \\ (a, b) \otimes c &\longmapsto \int_M a \wedge c + \int_N b \wedge c, \end{aligned}$$

which induces a pairing on the corresponding homology and cohomology.

1.3. Goresky-MacPherson-Siegel $\mathcal{L}$ -class. Recall that [Si, p.1068] a pseudo-manifold is a Witt-space if every even dimensional link $\mathbf{L}^\ell$ of an odd codimensional intrinsic stratum satisfies $IH_{\frac{\ell}{2}}^{\bar{m}}(\mathbf{L}; \mathbb{Q}) = 0$. Let V be a Witt-space which is also a compact subset of some C^∞ -manifold V' . If $\dim V \neq 4k$, then we define the signature σ of V to be 0. If $\dim V = 4k$, then let $\bar{m} = (0, 0, 1, 1, 2, 2, \dots, 2k-1)$. By [Si, Theorem 3.4], there is a non-degenerate rational pairing

$$IH_i^{\bar{m}}(V; \mathbb{Q}) \times IH_j^{\bar{m}}(V; \mathbb{Q}) \rightarrow \mathbb{Q}$$

for $i + j = \dim V$, $i, j \geq 0$. In this case, the signature σ of V is defined as the signature of the associated quadratic form.

As in [GM, p.158], a continuous map $f : V \rightarrow S^k$ is called transverse if

- (a): f is the restriction of a C^∞ -map $\tilde{f} : U \rightarrow S^k$ for some neighborhood U of V in V' ,
- (b): $\tilde{f}$ is transverse regular to the north pole $p \in S^k$,
- (c): $\tilde{f}^{-1}(p)$ is transverse to each stratum of V .

As there is such a representative in each homotopy class and the signature is cobordism invariant [GM, p.158], [Si, Theorem 2.1], one can define

$$\begin{aligned} \theta : [V, S^k] &\longrightarrow \mathbb{Z}, \\ [f] &\longmapsto \sigma(f^{-1}(p)). \end{aligned}$$

The Goresky-MacPherson-Siegel $\mathcal{L}$ -class, $\mathcal{L}_k(V) \in H_k(V; \mathbb{R})$, is defined as the homomorphism

$$\theta \otimes I : H^k(V; \mathbb{R}) \rightarrow \mathbb{R}$$

where we have identified $[V, S^k] \otimes \mathbb{R} \cong H^k(V; \mathbb{R})$ when $2k > m + 1$.

We can remove the assumption $2k > m + 1$ by crossing V with a sphere as in [GM, p.158] and [MS].

2. SIGNATURE OPERATOR AND K -CYCLE

2.1. Finite propagation speed. Let Y be an admissible Riemannian pseudo-manifold. In this subsection, the domains of the operators are:

$$\begin{aligned}\text{Dom}(d) &= \{\alpha \in \Gamma^\infty : \alpha, d\alpha \in L^2\}, \\ \text{Dom}(\delta) &= \{\alpha \in \Gamma^\infty : \alpha, \delta\alpha \in L^2\}.\end{aligned}$$

In the next subsection, we will show that it does not matter which domain we use.

By [C1], $\bar{d}^* = \bar{\delta}$. Then by [H1, Lemma 4.3], $D := \bar{d} + \bar{\delta}$ is self-adjoint with domain $\text{Dom}(\bar{d}) \cap \text{Dom}(\bar{\delta})$.

Let $f \in C(Y)$ act on $L^2(\wedge^* T(Y \setminus \sum))$ by multiplication.

Lemma 2.1. *Let $h \in C^{Lip}(Y)$; then $h \cdot \text{Dom}(D) \subset \text{Dom}(D)$.*

Proof. Let $\omega \in \text{Dom}(D)$ and J_ϵ be the mollifier corresponding to a bump function Φ . So $J_\epsilon(h\omega)$ is smooth and $J_\epsilon(h\omega) \xrightarrow{L^2} h\omega$. Since

$$\begin{aligned}d(h\omega) &= dh \wedge \omega + h d\omega, \\ \delta(h\omega) &= *(dh \wedge *\omega) + h \delta\omega,\end{aligned}$$

$\|f\|_\infty, \|dh\|_\infty \leq 1$, $\omega, d\omega, \delta\omega \in L^2$ and by using partition of unity, we have

$$\begin{aligned}J_\epsilon(h\omega)(x) &= \frac{1}{\epsilon^n} \int \Phi\left(\frac{x-y}{\epsilon}\right) h(y) \omega(y) dy \\ &= \frac{1}{\epsilon^n} \int \Phi\left(\frac{y}{\epsilon}\right) h(x-y) \omega(x-y) dy\end{aligned}$$

in local coordinate patch.

Then $J_\epsilon(h\omega), d(J_\epsilon(h\omega)), \delta(J_\epsilon(h\omega)), d(h\omega), \delta(h\omega) \in L^2$.

By Friedrichs Lemma [T, p.114],

$$dJ_\epsilon(h\omega) \xrightarrow{L^2} d(h\omega), \quad \delta J_\epsilon(h\omega) \xrightarrow{L^2} \delta(h\omega).$$

Hence $h\omega \in \text{Dom}(D)$. □

Proposition 2.2. *D has finite propagation speed with respect to $C(Y)$, i.e.*

$$\forall t \in \mathbb{R}, \quad \text{supp}(e^{itD}) \subset \{(x, y) \in Y \times Y : \rho_Y(x, y) \leq |t|\}.$$

Also, let $f \in \mathcal{S}(\mathbb{R})$ such that $\text{supp} \hat{f} \subset [-\alpha, \alpha]$ for some $\alpha > 0$; then

$$\text{supp}(f(D)) \subset \{(x, y) \in Y \times Y : \rho_Y(x, y) \leq \alpha\}.$$

Proof. As ρ_Y is a Lipschitz function on $Y \setminus \sum^{m-2}$, the results follow from [H2, Lemma 1.10] as in [H2, Corollary 1.11]. □

In the remainder of this paper, we will assume m to be even, unless otherwise stated.

Remark 2.3. So far, the results for Y in this section are still true when Y endows a metric which is quasi-isometric to a flat metric.

2.2. Essential self-adjointness of signature operator. Due to some technicalities in the rest of this paper, we will investigate the essential self-adjointness of the twisted signature operator on $X^\dagger$. Assume $\dim X$ is even; then $\partial X = L \times N$ is odd dimensional. Let (E, ∇) be a Hermitian vector bundle on X with a unitary connection such that its restriction to $c(L) \times N$ is pulled back from N .

We will consider the scaling of the metric in the conical direction as follows:

$$g|_{c(L) \times N} = \left(\frac{dr^2}{\epsilon} + \psi(r)^2 g^L \right) \oplus g^N$$

for $\epsilon > 0$.

In the following we will use the standard domain, specifically $\text{Dom}(D) = \{\alpha \in \Gamma_c^\infty(X)\}$.

To study the self-adjointness of a operator D , we will examine the deficiency indices $n_\pm(D) = \dim \text{Ker}(D \mp iI)$. Let us recall a proposition from [L1].

Proposition 2.4 ([L1], Corollary 2.2). *Let $\mathcal{M}_i$ be an oriented Riemannian manifold and D_i be a generalized Dirac operator over $\mathcal{M}_i$, $i = 1, 2$. Let U_i be an open subset of $\mathcal{M}_i$, $i = 1, 2$. Suppose $\mathcal{M}_1 \setminus U_1$ and $\mathcal{M}_2 \setminus U_2$ are complete manifolds with compact boundary and there exists an isometry from $\gamma : U_1 \rightarrow U_2$ which lifts to an isomorphism of Clifford structure. Then $n_\pm(D_1) = n_\pm(D_2)$.*

For easy reference, we will add subscripts to operators to indicate the underlying manifold and vector bundle.

Proposition 2.5. *There exists $\delta > 0$ (independent of E) such that $\forall \epsilon \in (0, \delta)$, the twisted signature operator $D_{X,E}$ is essentially self-adjoint.*

Proof. We will divide this into two cases.

Case 1: ℓ is odd and n is even.

On $c(L) \times N$, $D_{X,E} \simeq D_{c(L)} \hat{\otimes} I + I \hat{\otimes} D_{N,E}$.

For sufficiently small ϵ , by [BS, Lemma 5.4], $D_{c_{0,\infty}(L)}$ is essentially self-adjoint. Hence, by [RS, Theorem VIII.33], $D_{c_{0,\infty}(L) \times N, E}$ is essentially self-adjoint. Therefore,

$$n_\pm(D_{c_{0,\infty}(L) \times N, E}) = 0.$$

Then by Proposition 2.4,

$$n_\pm(D_{X,E}) = n_\pm(D_{c_{0,\infty}(L) \times N, E}) = 0.$$

Therefore, $D_{X,E}$ is essentially self-adjoint for sufficiently small ϵ .

Case 2: ℓ is even and n is odd.

Since $\dim(c_{0,\infty}(L))$ is odd, the signature operator splits as

$$D_{c_{0,\infty}(L)} = D_{c_{0,\infty}(L)}^+ \oplus D_{c_{0,\infty}(L)}^-.$$

Let λ_{ω_L} be the Clifford multiplication by $i^{\lfloor \frac{\ell+1}{2} \rfloor} e_1 \cdots e_\ell$ where $(e_1, \dots, e_\ell)$ is an oriented orthonormal frame for TL .

By [L1, Lemma 1.2, Proposition 4.1, Proposition 5.3],

$$\begin{aligned} n_\pm(D_{c_{0,\infty}(L)}^\pm) &= n_\mp(D_{c_{0,\infty}(L)}^\mp) \\ &= \dim(\text{Ker}(\lambda_{\omega_L} \mp I) \cap \text{Ker} P) + \sum_{0 < \lambda < \frac{1}{2}} \dim \text{Ker}(P - \lambda I) \end{aligned}$$

for some operator P .

By [BL, Lemma 2.2], $\text{Ker} P = H^{\frac{\epsilon}{2}}(L) = 0$.

By [BL, Corollary 2.3],

$$\sum_{0 < \lambda < \frac{1}{2}} \dim \text{Ker}(P - \lambda I) = 0 \quad \text{for sufficiently small } \epsilon.$$

Thus, $D_{c_0, \infty}^{\pm}(L)$ is essentially self-adjoint and so is $D_{c_0, \infty}(L)$. Notice that

$$D_{c_0, \infty}(L) \times N, E = D_{c_0, \infty}(L) \otimes I + \phi \otimes D_{N, E}, \quad \text{where } \phi = \begin{cases} 1 & \text{on } \Omega^{\text{even}}, \\ -1 & \text{on } \Omega^{\text{odd}}. \end{cases}$$

Then as in the proof of [RS, Theorem VIII.33], $D_{c_0, \infty}(L) \times N, E$ is essentially self-adjoint. Thus,

$$n_{\pm}(D_{c_0, \infty}(L) \times N, E) = 0.$$

By Proposition 2.4,

$$n_{\pm}(D_{X, E}) = n_{\pm}(D_{c_0, \infty}(L) \times N, E) = 0.$$

Hence the result follows. $\square$

2.3. Singular elliptic estimate. In the remaining of this paper, we will assume a scaling on the conical metric such that there are unique self-adjoint extensions for the signature operator on X and the twisted signature operator on X as well as the twisted signature operator on $c_0, \infty(L) \times N$. These facts will be used to remove the effect of small eigenvalues. By abuse of notation, we will use the same symbol to denote the self-adjoint extensions of the operators. As in Section 2.2, $\dim X$ is even.

Let (E, ∇) be a Hermitian vector bundle on X with unitary connection such that its restriction to $c(L) \times N$ is pulled back from N .

To furnish the computation on heat kernels, we need to recall a definition from [L2]. Let $\mathcal{M}$ be a Riemannian manifold and $U \subset \mathcal{M}$ an open subset with compact boundary. Let P_0 be a symmetric differential operator of order μ defined on a bundle over $\mathcal{M}$. Assume there exists a closed self-adjoint extension P of P_0 such that $\text{Dom}(P)$ is invariant under multiplication by functions $\varphi \in C_U^{\infty}(M)$ satisfying $\varphi|_U \equiv 1$. (This is always true if $\mathcal{M} \setminus U$ is compact.)

Let $\mathcal{KD}(P, U) := \{s \in \text{Dom}(P) : \text{supp } s \subset U, \text{dist}(\text{supp } s, \partial U) > 0\}$.

Definition 2.6 ([L2], p.41). P satisfies the singular elliptic estimate (SE) on U if $\exists \varrho \in L_{loc}^2(\mathcal{M}) \cap C(\mathcal{M})$, $\varrho > 0$, $\varrho|_U \in L^2(U)$ and $\ell \in \mathbb{R}^+$ such that for $x \in U$ and $s \in \mathcal{KD}(P^{\ell}, U)$,

$$|s(x)| \leq \varrho(x) (\|s\|_{L^2(U, E)} + \|P^{\ell} s\|_{L^2(U, E)}).$$

The importance of this concept lies in the following theorem.

Theorem 2.7 ([L2]). Let $\mathcal{M}_i, U_i, P_{i,0}$ and P_i , $i = 1, 2$ as above. Assume there is an isometry $F : U_1 \rightarrow U_2$, which lifts to a bundle isometry $F_* : E_1|_{U_1} \rightarrow E_1|_{U_2}$ such that $P_{1,0} = F_*^{-1} \circ P_{2,0} \circ F_*$. We will identify U_1 with U_2 and denote it by U . We choose an open subset $W \subset U$, with smooth compact boundary such that $\overline{W} \subset U$ and $U \setminus W$ is relatively compact. If P_1 and P_2 satisfy (SE) over W , then $\forall N > 0, \exists C > 0$ such that for $x, y \in W$,

$$\left| \left((P_1^2)^k e^{-tP_1^2} \right) (x, y) - \left((P_2^2)^k e^{-tP_2^2} \right) (x, y) \right| \leq C \varrho(x) \varrho(y) t^N$$

where $k \in \mathbb{Z}_+ \cup \{0\}$.

In order to establish singular elliptic estimate for X , we need to recall some notation from [L2].

Let $\tilde{\rho} : X \rightarrow \mathbb{R}$ be a smooth function such that

$$\begin{aligned} \tilde{\rho}|_M &= 1, \\ \tilde{\rho}((r, x, y)) &= r \quad \text{on } c_{0, \frac{1}{2}}(L) \times N. \end{aligned}$$

Let $\mathcal{H}$ be a Hilbert space, $T > 0$ be a self-adjoint operator on $\mathcal{H}$ and

$$\mathcal{D}^\infty(T) := \bigcap_{k \geq 1} \text{Dom}(T^k).$$

For $x, y \in \mathcal{D}^\infty(T)$ and $s \in \mathbb{R}$,

$$(x, y)_s := (T^s x, T^s y).$$

Let $\mathcal{H}_T^s$ be the completion of $\mathcal{D}^\infty(T)$ with respect to $\|\cdot\|_s$. Then by [L2, Lemma 1.2.1], $\Delta_L = (r \frac{\partial}{\partial r})^t (r \frac{\partial}{\partial r}) + D_L^2 \geq \frac{1}{4}$ and $\Delta = \Delta_L \otimes I + I \otimes D_{N,E}^2$ is essentially self-adjoint on $C_c^\infty(c_{0,\infty}(L)) \otimes C^\infty(N, E)$.

Define

$$\begin{aligned} \mathcal{H}^{s,0}(c_{0,\infty}(L) \times N, E) &:= \mathcal{H}_{\Delta}^{\frac{s}{2}}, \\ \mathcal{H}^{s,\gamma}(c_{0,\infty}(L) \times N, E) &:= r^\gamma \mathcal{H}^{s,0}(c_{0,\infty}(L) \times N, E) \\ &\quad \text{with scalar product } (f, g)_{s,\gamma} := (r^{-\gamma} f, r^{-\gamma} g)_s. \end{aligned}$$

Now let $\Delta_{X,E}$ be a non-negative elliptic differential operator of order 2 on X such that

$$\begin{aligned} \Delta_{X,E}|_M &\geq c_1 && \text{for some } c_1 > 0 \\ \text{and} \quad \Delta_{X,E}|_{c(L) \times N} &= \Delta. \end{aligned}$$

By [L2, Corollary 2.2], $\Delta_{X,E}$ is essentially self-adjoint on $C_c^\infty(X, E)$.

Define

$$\begin{aligned} \mathcal{K}^{s,0}(X, E) &:= \mathcal{H}_{\Delta_{X,E}}^{\frac{s}{2}}, \\ \mathcal{K}^{s,\gamma}(X, E) &:= \tilde{\rho}^\gamma \mathcal{K}^{s,0}(X, E) \\ &\quad \text{with scalar product } (f, g)_{s,\gamma} := (\tilde{\rho}^{-\gamma} f, \tilde{\rho}^{-\gamma} g)_s. \end{aligned}$$

In [L2] Lesch proved the following lemma in the case when N is equal to a point. We shall prove this in a more general context.

Lemma 2.8. *There exist constants C and $\mu > 0$ such that*

$$\text{Dom}(D_{X,E}^m) \subset \mathcal{K}^{m,\mu}(X, E)$$

and $\forall f \in \text{Dom}(D_{X,E}^m)$,

$$\|f\|_{m,\mu} \leq C (\|f\|_{0,0} + \|D_{X,E}^m f\|_{0,0}).$$

Proof. Let $\varphi \in C_c^\infty(\mathbb{R})$ such that $\varphi \equiv 1$ near 0.

By [L2, Proposition 1.3.19], the case when N is equal to a point, there exists $\mu > 0$ such that $\varphi \text{Dom}(D_{c(L)}^m) \subset \mathcal{H}^{m,\mu}(c_{0,\infty}(L))$. So,

$$\varphi \text{Dom}(D_{c(L)}^m) \otimes \text{Dom}(D_{N,E}^m) \subset \mathcal{H}^{m,\mu}(c_{0,\infty}(L) \otimes N, E).$$

Hence, $\text{Dom}(D_{X,E}^m) \subset \mathcal{K}^{m,\mu}(X, E)$.

The inequality in the lemma is now equivalent to the assertion that $\text{Dom}(D_{X,E}^m) \hookrightarrow \mathcal{K}^{m,\mu}(X, E)$ is continuous. This follows from Closed Graph Theorem and the fact that the Sobolev norms in $\text{Dom}(D_{X,E}^m)$ and $\mathcal{K}^{m,\mu}(X, E)$ are stronger than the L^2 -norm. $\square$

Proposition 2.9. *There exist constants C and $\mu > 0$ such that $\forall f \in \text{Dom}(D_{X,E}^m)$,*

$$|f(x)| \leq C\bar{\rho}(x)^{\mu-\frac{1}{2}}(\|f\| + \|D_{X,E}^m f\|).$$

That is, $D_{X,E}$ satisfies the singular elliptic estimate (Definition 2.6).

Proof. Follows from Lemma 2.8 and the corresponding estimate in model cone [L2, Cor. 1.2.9]. $\square$

2.4. The construction of K -cycle. Let $\mathcal{H} = L^2(\wedge^* T(Y \setminus \Sigma)) = \mathcal{H}^+ \oplus \mathcal{H}^-$ with grading induced by $\epsilon = i^{p(p-1)+\frac{m}{2}} *$ and $*\text{Dom}(D) \subset \text{Dom}(D)$, $D = \begin{pmatrix} 0 & D^- \\ D^+ & 0 \end{pmatrix}$.

$A = C(Y) \xrightarrow{\text{dense}} \mathfrak{A} = C^{\text{Lip}}(Y)$. Clearly $\mathfrak{A}$ is closed under holomorphic calculus of A and, by Lemma 2.1, leaves $\text{Dom}(D)$ invariant and $\|[D, a]\| < \infty, \forall a \in \mathfrak{A}$.

Proposition 2.10. *($\mathcal{H}, D$) is an unbounded p -summable Fredholm module for $p > m$. Similar result holds for $X^\dagger$.*

Proof. By [G], $\overline{D} = \bar{d} \bar{d}^* + \bar{\delta} \bar{\delta}^* = D^2$ is self-adjoint. By [C2, Lemma 7.1], D^2 has a discrete spectrum with finite multiplicities. By [C2, Theorem 7.2], we have the following asymptotic expansion of the heat kernel:

$$\text{Tr}(e^{-tD^2}) = \int_X \text{tr} e^{-tD^2}(x, x) \sim c_m + \sum_{j=0}^{m-2} c_j t^{-\frac{m}{2} + \frac{j}{2}} + O(t^{-\frac{1}{2}}).$$

$$\text{i.e.} \quad \int_X \text{tr} e^{-tD^2}(x, x) \sim t^{-\frac{m}{2}} c_0 + O(t^{-\frac{m-1}{2}}).$$

By Karamata Theorem [BGV, p.94],

$$\begin{aligned} N(\lambda) &\sim \frac{c_0}{\Gamma(\frac{m+2}{2})} \lambda^{\frac{m}{2}} \\ \lambda_j &\sim C j^{\frac{2}{m}} \\ \text{Tr}((1 + D^2)^{-\frac{p}{2}}) &< \infty \quad \text{for } p > m. \end{aligned}$$

By Proposition 2.9 and Theorem 2.7, one can obtain the asymptotic expansion on $X^\dagger$. Then result for $X^\dagger$ follows as above. $\square$

3. “STRAIGHT” CHERN CHARACTER AND $\mathcal{L}$ -CLASS

3.1. “Straight” Chern character. Since we have a Fredholm module for the signature operator, we can repeat the construction of the “straight” Chern character as in [CM] and [MW1]. Let’s recall their construction:

Let $\bar{u}$ be an even smooth function on $\mathbb{R}$ such that $\bar{v}(x) = 1 - x^2 \bar{u}(x)$ is Schwartz and both $\bar{u}$ and $\bar{v}$ have Fourier transforms supported on $(-\frac{1}{4}, \frac{1}{4})$. As $\bar{u}$ and $\bar{v}$ are even,

$$\bar{u}(x) = u(x^2), \quad \bar{v}(x) = v(x^2)$$

for some smooth functions u and v .

Clearly v is also Schwartz and so is

$$w(x) = v(x)(1 + v(x))u(x).$$

Let $\gamma = i^{\lfloor \frac{m+1}{2} \rfloor} (-1)^{p(m-p) + \frac{p(p+1)}{2}} *_p : L_2(\Omega^p(X)) \rightarrow L_2(\Omega^{m-p}(X))$.

Then we consider the idempotent

$$P(tD) = \begin{pmatrix} (v(t^2 D^2))^2 \gamma & w(t^2 D^2) \cdot tD\gamma \\ -v(t^2 D^2) \cdot tD\gamma & (v(t^2 D^2))^2 \gamma \end{pmatrix} + \begin{pmatrix} \frac{1-\gamma}{2} & 0 \\ 0 & \frac{1-\gamma}{2} \end{pmatrix}$$

and define an Alexander-Spanier cycle $\Lambda_*(tD)$ as follows:

Let $f^0, \dots, f^{2q} \in C(X^\dagger)$,

$$q = 0, \quad \Lambda_0(tD)(f^0) := \frac{1}{2} \text{Tr} \left(P(tD)f^0 - \begin{pmatrix} \frac{1-\gamma}{2} & 0 \\ 0 & \frac{1-\gamma}{2} \end{pmatrix} f^0 \right),$$

$$q > 0, \quad \Lambda_{2q}(tD)(f^0 \otimes \dots \otimes f^{2q})$$

$$:= \frac{(2\pi i)^q}{q!(2q+1)2} \text{Tr} \left(\sum_{\sigma \in S_{2q+1}} \text{sgn}(\sigma) P(tD)f^{\sigma(0)} \dots P(tD)f^{\sigma(2q)} \right),$$

and $\overline{ch}_{2q}(D) := \lim_{t \rightarrow 0} \Lambda_{2q}(tD)$.

By using the finite propagation speed, as in [MW1], we have

Proposition 3.1. a) *Given two isometric open embedding of admissible Riemannian pseudomanifolds $(U, \rho_U) \hookrightarrow (Y_i, \rho_{Y_i})$, $i = 1, 2$, and a compact subset K of U , there is a $\delta > 0$ such that $\forall t \in (0, \delta)$ and $f^0, \dots, f^k \in C_c^{Lip}(U)$ with at least one of the f^j 's having support inside K , one has*

$$\Lambda_k(P(tD_1))|_U(f^0 \otimes \dots \otimes f^k) = \Lambda_k(P(tD_2))|_U(f^0 \otimes \dots \otimes f^k),$$

where D_i is the signature operator of (Y_i, ρ_{Y_i}) as defined in Section 2.1.

b) *For any $2q > m = \dim Y$ and $f^i \in C^{Lip}(Y)$, one has*

$$\lim_{t \rightarrow 0} \Lambda_{2q}(tD)(f^0 \otimes \dots \otimes f^{2q}) = 0.$$

Moreover, $\Lambda_{2q}(tD) = \partial \int_0^t \nu_{2q+1}(sD)ds$ where ν is defined as in [MW1, Section 3.2].

c) *The results in (a) and (b) hold for $X^\dagger$.*

Proof. Same as [MW1, Theorem 2.2, Theorem 3.3]. $\square$

Proposition 3.2. *Let $\delta_\ell = \frac{1}{2}(1 - (-1)^\ell)$. For any $f^i \in C_{SA}^\infty(X^\dagger)$,*

$$\begin{aligned} & \lim_{t \rightarrow 0} \Lambda_{2q}(tD)(f^0 \otimes \dots \otimes f^{2q}) \\ &= \int_M 2^{\frac{m}{2}} \mathcal{L}(R(g^M)) f^0 df^1 \wedge \dots \wedge df^{2q} \\ & \quad - \delta_\ell \eta(L) 2^{\frac{n}{2}} \int_N \mathcal{L}(R(g^N)) f^0 df^1 \wedge \dots \wedge df^{2q}, \end{aligned}$$

where $\mathcal{L}(R(g^M))$ and $\mathcal{L}(R(g^N))$ are the Atiyah-Hirzebruch $\mathcal{L}$ -polynomials in the curvature of the Levi-Civita connection of the metrics g^M and g^N respectively.

Proof. Following [MW2], we choose $\rho_0 \in C^\infty([0, 1])$ with $\rho_0(r) \in [0, 1]$ such that

$$\rho_0(r) = \begin{cases} 1 & \forall r \in [0, \frac{1}{2}], \\ 0 & \forall r \in [\frac{3}{4}, 1], \end{cases}$$

and define $\rho : X^\dagger \rightarrow \mathbb{R}$ by

$$\rho(x) = \begin{cases} 0 & \text{if } x \in M, \\ \rho_0(r) & \text{if } x = (r, s, y) \in c(L) \times N. \end{cases}$$

Clearly $\rho \in C_{SA}^\infty(X^\dagger)$.

Let $\mathbf{f} = f^0 \otimes \cdots \otimes f^{2q}$. Write $\mathbf{f} = \mathbf{f}^{(1)} + \mathbf{f}^{(2)}$, where $\mathbf{f}^{(1)} := (\rho f^0) \otimes f^1 \cdots \otimes f^{2q}$ and $\mathbf{f}^{(2)} := ((1 - \rho)f^0) \otimes f^1 \cdots \otimes f^{2q}$.

$$(1) \quad \lim_{t \rightarrow 0} \langle \Lambda_{2q}(tD), \mathbf{f} \rangle = \lim_{t \rightarrow 0} \langle \Lambda_{2q}(tD), \mathbf{f}^{(1)} \rangle + \lim_{t \rightarrow 0} \langle \Lambda_{2q}(tD), \mathbf{f}^{(2)} \rangle.$$

By locality (Proposition 3.1) and [MW1, Theorem 3.4],

$$\begin{aligned} \lim_{t \rightarrow 0} \langle \Lambda_{2q}(tD), \mathbf{f}^{(2)} \rangle &= 2^{\frac{m}{2}} \int_M \mathcal{L}(R(g^M))(1 - \rho)f^0 df^1 \wedge \cdots \wedge df^{2q} \\ &= 2^{\frac{m}{2}} \int_M \mathcal{L}(R(g^M))f^0 df^1 \wedge \cdots \wedge df^{2q}. \end{aligned}$$

Case 1: ℓ is even and n is odd.

By locality (Proposition 3.1) and product formula [MW2, Proposition 5.1],

$$\lim_{t \rightarrow 0} \langle \Lambda_{2q}(tD), \mathbf{f}^{(1)} \rangle = 0.$$

Case 2: ℓ is odd and n is even.

As ℓ is odd, by Thom's Theorem [St, p.183], there exists a smooth, compact and oriented manifold W with $\partial W = 2L$.

Let $Y = (W \cup 2c(L)) \times N$,

$$\tilde{M} = 2M \cup (-W \times N),$$

$\pi_Y : Y \rightarrow N$ be the projection map onto the second factor,

and $\mathbf{f}_Y = \pi_Y^*(f^0 \otimes \cdots \otimes f^{2q}) \in C_{SA}^\infty(Y)$.

Let $(f)_2$ be the function on $2X$ which equals f on each copy of X . Now $(\mathbf{f}^{(1)})_2 = \mathbf{f}_Y^{(1)} = \mathbf{f}_Y - \mathbf{f}_Y^{(2)}$. By locality (Proposition 3.1),

$$\begin{aligned} 2\langle \Lambda_{2q}(tD_X), \mathbf{f}^{(1)} \rangle &= \langle \Lambda_{2q}(tD_{2X}), (\mathbf{f}^{(1)})_2 \rangle \\ &= \langle \Lambda_{2q}(tD_Y), \mathbf{f}_Y^{(1)} \rangle \\ (2) \quad &= \langle \Lambda_{2q}(tD_Y), \mathbf{f}_Y \rangle - \langle \Lambda_{2q}(tD_Y), \mathbf{f}_Y^{(2)} \rangle. \end{aligned}$$

By using the index theorem in [C2] instead of the McKean-Singer formula at the end of the proof in [MW2, Lemma 5.2], we see that we still have a product formula for space with conical singularities. Therefore,

$$\begin{aligned} \langle \Lambda_{2q}(tD_Y), \mathbf{f}_Y \rangle &= \text{sign}(W \cup 2c(L)) \cdot \langle \Lambda_{2q}(tD_N), f^0 \otimes \cdots \otimes f^{2q} \rangle \\ (3) \quad &= \text{sign}(W \cup 2c(L)) 2^{\frac{n}{2}} \int_N \mathcal{L}(R(g^N))f^0 df^1 \cdots df^{2q}. \end{aligned}$$

The last line follows from [MW1, Theorem 3.4]. Now,

$$2\langle \Lambda_{2q}(tD_X), \mathbf{f}^{(2)} \rangle - \langle \Lambda_{2q}(tD_Y), \mathbf{f}_Y^{(2)} \rangle = \langle \Lambda_{2q}(tD_A), \mathbf{f}_A^{(2)} \rangle,$$

where $\mathbf{f}_A^{(2)} = (\mathbf{f}^{(2)})_2 \sqcup \mathbf{f}_Y^{(2)}$ and $A = 2X \cup -Y = \tilde{M} \cup 2(c(L) \cup (-c(L))) \times N$.

Let $Z = (c(L) \cup (-c(L))) \times N$. Then $\mathbf{f}_A^{(2)} = \tilde{\mathbf{f}} \sqcup \mathbf{f}_Z^{(2)}$.

Clearly, $\langle \Lambda_{2q}(tD_Z), \mathbf{f}_Z^{(2)} \rangle = 0$. Then by locality (Proposition 3.1),

$$(4) \quad 2\langle \Lambda_{2q}(tD_X), \mathbf{f}^{(2)} \rangle - \langle \Lambda_{2q}(tD_Y), \mathbf{f}_Y^{(2)} \rangle = \langle \Lambda_{2q}(tD_{\tilde{M}}), \tilde{\mathbf{f}} \rangle.$$

Together with (1), (2), (3) and (4), we have

$$\begin{aligned} & 2\langle \lim_{t \rightarrow 0} \Lambda_{2q}(tD), \mathbf{f} \rangle \\ &= \lim_{t \rightarrow 0} \left(\langle \Lambda_{2q}(tD_Y), \mathbf{f}_Y \rangle - \langle \Lambda_{2q}(tD_Y), \mathbf{f}_Y^{(2)} \rangle + 2\langle \Lambda_{2q}(tD_X), \mathbf{f}^{(2)} \rangle \right) \\ &= \text{sign}(W \cup 2c(L)) \, 2^{\frac{n}{2}} \int_N \mathcal{L}(R(g^N)) f^0 df^1 \cdots df^{2q} + \lim_{t \rightarrow 0} \langle \Lambda_{2q}(tD_{\tilde{M}}), \tilde{\mathbf{f}} \rangle. \end{aligned}$$

As $\tilde{M}$ is smooth, by [MW1, Theorem 3.4] and the product formula [MW1, Proposition 5.1], we have

$$\begin{aligned} & \lim_{t \rightarrow 0} \langle \Lambda_{2q}(tD_{\tilde{M}}), \tilde{\mathbf{f}} \rangle \\ &= 2 \int_M 2^{\frac{n}{2}} \mathcal{L}(R(g^M)) f^0 df^1 \cdots df^{2q} \\ & \quad - \int_W 2^{\frac{\ell+1}{2}} \mathcal{L}(R(g^W)) \cdot \int_N 2^{\frac{n}{2}} \mathcal{L}(R(g^N)) f^0 df^1 \cdots df^{2q}. \end{aligned}$$

But by [C2],

$$\text{sign}(W \cup 2c(L)) = \int_W 2^{\frac{\ell+1}{2}} \mathcal{L}(R(g^W)) - 2\eta(L).$$

Hence the result follows. $\square$

3.2. Analytic realization of Goresky-MacPherson-Siegel $\mathcal{L}$ -class. In this subsection, we will show that, under a condition on the link, the “straight” Chern character and the Goresky-MacPherson-Siegel $\mathcal{L}$ -class are the same (up to constants). This gives an analytic realization of $\mathcal{L}$ -class.

In the following, we will denote the $\mathcal{L}$ -class as an element in $H_k(X^\dagger)$ and $\overline{H}_k(X^\dagger)$ by the same symbol.

Theorem 3.3. *If the space $X^\dagger$ is even dimensional and $2L$ is oriented cobordant to zero, then*

$$[\overline{ch}_{2q}(D)] = [2^q \mathcal{L}_{2q}(X^\dagger)].$$

Proof. Notice that any element in $[X^\dagger, S^{2q}]$ is homotopic to an element in

$$\begin{aligned} C_{SA}^\infty(X^\dagger, S^{2q}) &= \{f \in C(X^\dagger, S^{2q}) : \text{smooth on } X \text{ such that} \\ & \quad f|_{c(L) \times N} = pr^*(g) \text{ for } g \in C^\infty(N, S^{2q})\}. \end{aligned}$$

By [GM], it suffices to show that for any $f \in C_{SA}^\infty(X^\dagger, S^{2q})$ which is transverse,

$$\langle \overline{ch}_{2q}(D), f^*(s^{2q}) \rangle = 2^q \sigma(f^{-1}(p)),$$

where σ is the signature in Section 1.3 and s^{2q} is a fixed generator of $\overline{H}^k(S^{2q})$. By [BT, p.37], s^{2q} and $f^*(s^{2q})$ are sums of elementary tensors. Note that the normalization constants of $\mathcal{L}$ -classes are different from [MS].

Then $f|_{c(L) \times N} = (pr)^*(g)$ for some $g : N \rightarrow S^{2q}$ which is smooth and transverse at p . In the rest of this proof, we will adopt the notation used in the proof of Proposition 3.2.

As in the proof of Proposition 3.2, we have

$$\begin{aligned}
& \langle \overline{ch}_{2q}(D), f^*(s^{2q}) \rangle \\
&= \frac{1}{2} \left(\text{sign}(W \cup 2c(L)) \langle \overline{ch}_{2q}(D_N), g^*(s^{2q}) \rangle + \langle \overline{ch}_{2q}(D_{\tilde{M}}), \tilde{f}^*(s^{2q}) \rangle \right) \\
&= \frac{2^q}{2} \left[\sigma(W \cup 2c(L)) \sigma(g^{-1}(p)) + \sigma(2(f^{-1}(p) \cap M) \cup (-W \times g^{-1}(p))) \right] \\
&\quad \text{[MW1], [MS]} \\
&= \frac{2^q}{2} \left[\sigma((W \cup 2c(L)) \times g^{-1}(p)) + \sigma(2(f^{-1}(p) \cap M) \cup (-W \times g^{-1}(p))) \right] \\
&= 2^q \sigma((c(L) \times g^{-1}(p)) \cup (f^{-1}(p) \cap M)) \\
&= 2^q \sigma(f^{-1}(p)).
\end{aligned}$$

The last three lines follow from the multiplicativity and additivity of σ [Si]. $\square$

Remark 3.4. 1. If the space $X^\dagger$ is even dimensional, then $[\mathcal{L}_{2q+1}(X^\dagger)] = [0]$.

2. By Thom's Theorem [St, p.183], $2L$ is oriented cobordant to zero iff all Pontryagin numbers are zero. It is always true if $\ell \not\equiv 0 \pmod{4}$.

Theorem 3.5. *For any admissible pseudomanifold $X^\dagger$ with one singular stratum such that $2L$ is oriented cobordant to zero, one has*

$$L_*(X^\dagger; \rho_{X^\dagger}) := 2^{2m'} \mathcal{L}_{m'}(R(g^M)) \oplus -\delta_\ell \eta(L) 2^{2n'} \mathcal{L}_{n'}(R(g^N)) \in \Omega_*(X^\dagger)_{SA}$$

is a cycle which represents the $\mathcal{L}$ -class of $X^\dagger$, where $m' = \frac{m-*}{4}$, $n' = \frac{n-*}{4}$, $\delta_\ell = \frac{1}{2}(1 - (-1)^\ell)$ and $\mathcal{L}_{k'}(R(\cdot)) := 0$ for $k' \notin \mathbb{Z}$.

Proof. We will divide this into two cases.

Case 1: $\dim X$ is even.

It follows from the previous theorem (cf. [MW2, Theorem 4.3]).

Case 2: $\dim X$ is odd.

Let q_1, q_2 be the projections onto the first and second factors of $X^\dagger \times S^1$ respectively. Then by definition,

$$\mathcal{L}_k(X^\dagger)(\omega) = \mathcal{L}_{k+1}(X^\dagger \times S^1)(q_1^* \omega \wedge q_2^* s^1),$$

$$\text{where } \omega \in H^k(X^\dagger) \text{ and } H^1(S^1) = \langle s^1 \rangle.$$

The result follows from the definition of the pairing of Sullivan complexes (Section 1.2) and Case 1. $\square$

4. INDEX THEOREM

In this section, we will establish the index theorem for the pseudomanifolds with one singular stratum. And then we will identify the “straight” Chern character with the K-homology Chern character.

Theorem 4.1. *If the space $X^\dagger$ is even dimensional, then*

$$\text{Ind}(D_E) = 2^{\frac{m}{2}} \int_M \mathcal{L}(R(g^M)) \wedge ch(E) - \delta_\ell \eta(L) 2^{\frac{n}{2}} \int_N \mathcal{L}(R(g^N)) \wedge ch(E).$$

Proof. By Proposition 2.9 and Theorem 2.7, we have

$$\text{tr}_s e^{-tD_E^2}(x, x) = \int_{c(L) \times N} \text{tr}_s e^{-t\tilde{D}_E^2}(x, x) + O(t),$$

where $\tilde{D}_E$ is the twisted signature operator on $c_{0,\infty}(L) \times N$.

$$\text{Hence, } \text{Ind}(D_E) = \lim_{t \rightarrow 0} \left(\int_M \text{tr}_s e^{-tD_E^2}(x, x) + \int_{c(L) \times N} \text{tr}_s e^{-t\tilde{D}_E^2}(x, x) \right).$$

1° ℓ is odd and n is even.

Notice that $\tilde{D}_E = D_{c_0, \infty} \hat{\otimes} I + I \hat{\otimes} D_{N, E}$. Therefore, $\text{tr}_s e^{-t\tilde{D}_E^2}(x, x) = \text{tr}_s e^{-tD_{c(L)}^2}(x, x) \text{tr}_s e^{-tD_{N, E}^2}(x, x)$. By [C2],

$$\int_{c(L) \times N} \text{tr}_s e^{-t\tilde{D}_E^2}(x, x) = -\eta(L) \int_N 2^{\frac{n}{2}} \mathcal{L}(R(g^N)) \wedge ch(E).$$

2° ℓ is even and n is odd.

Note that $\tilde{D} = D_{c(L)} \otimes I + \phi \otimes D_{N, E}$ where $\phi = (-1)^k$ on $L^2(\Omega^k(c(L)))$. Also, $\gamma = \frac{1}{i} \gamma_1 \gamma_2$, $\gamma_1 = i^{\frac{\ell+2}{2}} c(vol_{c(L)}) \otimes I$ and $\gamma_2 = i^{\frac{n+1}{2}} \phi \otimes c(vol_N)$ where $c(vol_{c(L)})$ and $c(vol_N)$ are Clifford multiplication by volume elements in $c(L)$ and N respectively. Then,

$$\begin{aligned} \text{tr}_s e^{-t\tilde{D}_E^2}(x, x) &= \frac{1}{i} \text{tr}(\gamma_1 \gamma_2 e^{-tD_{c(L)}^2} \otimes e^{-tD_{N, E}^2})(x, x) \\ &= \frac{1}{i} \text{tr}(\gamma_1 e^{-tD_{c(L)}^2} \otimes e^{-tD_{N, E}^2} \gamma_2)(x, x) \\ &= \frac{1}{i} \text{tr}(\gamma_2 \gamma_1 e^{-tD_{c(L)}^2} \otimes e^{-tD_{N, E}^2})(x, x) \\ &= \frac{1}{i} \text{tr}(-\gamma_1 \gamma_2 e^{-tD_{c(L)}^2} \otimes e^{-tD_{N, E}^2})(x, x) \end{aligned}$$

Hence the result follows. $\square$

Remark 4.2. The above theorem is a special case of the index theorem announced in [BC]. They considered the index theory of twisted signature operators for fibration with fibers having conical singularities.

Corollary 4.3. *If the space $X^\dagger$ is even dimensional, then*

$$[ch(D)] = [\overline{ch}(D)].$$

Proof. By [MS, p.196 and Theorem 9.1], one can see that

$$ch(E_X) = ch(E_M) \oplus ch(E_N) \in H^*(\Omega^*(X^\dagger)_{SA}).$$

Thus, by Theorem 4.1, $\text{Ind}(D_E) = \langle 2^{\frac{n}{2}} L_*(X^\dagger; \rho_{X^\dagger}), ch(E) \rangle$. Hence the result follows from Theorem 3.3. $\square$

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