WEIGHTED WEAK-TYPE INEQUALITIES FOR THE MAXIMAL FUNCTION OF NONNEGATIVE INTEGRAL TRANSFORMS OVER APPROACH REGIONS

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ABSTRACT. The relation between approach regions and singularities of non-negative kernels $K_t(x, y)$ is studied, where $t \in (0, \infty)$, $x, y \in X$, and X is a homogeneous space. For $1 \leq p < q < \infty$, a sufficient condition on approach regions Ω_a ($a \in X$) is given so that the maximal function

$$\sup_{(x,t) \in \Omega_a} \int_X K_t(x, y) f(y) d\sigma(y)$$

is weak-type (p, q) with respect to a pair of measures σ and ω . It is shown that this condition is also necessary for operators of potential type in the sense of Sawyer and Wheedon (Amer. J. Math. **114** (1992), 813–874).

1. INTRODUCTION

In [6], A. Nagel and E. Stein improved the classical theorem of Fatou on nontangential limits of harmonic functions defined on $\mathbb{R}_+^{n+1}$ to include limits within regions which allow sequential approach with any degree of tangency to the boundary, $\mathbb{R}^n$. Their proof is based on a remarkable result of a necessary and sufficient condition on the approach regions so that the associated maximal function of Hardy-Littlewood type is weak type $(1, 1)$. The condition gives a clear picture of the relation between the size of cross-sectional measure of approach regions at height r and the “singularity”, r^{-n} , of the maximal operator of Hardy-Littlewood type. Following this important work, a series of papers has been devoted to maximal functions of Hardy-Littlewood type for associated approach regions (see, e.g. [1], [9], [5], and [7]). In this paper, we show that a similar relation holds for general integral transforms with nonnegative kernels in the upper half-space of a homogeneous space. We give a sufficient condition, which turns out to be necessary for operators of potential type in the sense of [8], on approach regions so that weak-type (p, q) ($1 \leq p < q < \infty$) inequalities (see (1.6) below) hold for the maximal function over the approach regions with respect to a pair of Borel measures σ and ω on X . Our condition and proof are motivated by a recent paper [3], where some general weak-type inequalities for integral transforms with nonnegative kernels are studied.

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A homogeneous space (X, d, μ) is a set X together with a quasi-metric d and a doubling measure μ . By the quasi-metric d we mean a mapping $d: X \times X \rightarrow [0, \infty)$ which satisfies the same conditions as a metric, except the triangle inequality is weakened to

$$(1.1) \quad d(x, y) \leq \kappa(d(x, z) + d(z, y)) \quad \text{for all } x, y, z \in X,$$

where $\kappa \geq 1$ is a constant which is independent of x, y , and z . Without loss of generality, we may assume that all balls $B(x, r) = \{y \in X: d(x, y) < r\}$ are open (see [4]). We also use the convention that $B(x, 0) = \emptyset$. By the doubling measure μ we mean a nonnegative measure on the Borel subsets of X so that $|2B|_\mu \leq C_\mu |B|_\mu$ for all balls $B \subset X$, where $|B|_\mu$ denotes the μ -measure of the ball B , and C_μ is a constant which depends only on μ .

For a homogeneous space X , we denote that $\hat{X} = X \times (0, \infty)$ the upper half-space of $X \times \mathbb{R}$, and that $\hat{B}(x, r) = B(x, r) \times (0, r)$. For a Borel measure σ on X , we study the operator T defined by

$$(1.2) \quad T(f d\sigma)(x, t) = \int_X K_t(x, y) f(y) d\sigma(y), \quad (x, t) \in \hat{X},$$

with the nonnegative kernel $K_t(x, y)$ which is lower semi-continuous.

Let $\Omega = \{\Omega_a: a \in X\}$ be a family of nonempty subsets in $\hat{X}$. For each $a \in X$, set $\Omega_a(r) = \{x \in X: (x, r) \in \Omega_a\}$ (the cross-section of Ω_a at height r). Following [9], for $\alpha > 0$ and $(x, r) \in \hat{X}$, we define

$$(1.3) \quad S_\alpha(x, r) = \{a \in X: \Omega_a(r) \cap B(x, \alpha r) \neq \emptyset\}.$$

We suppose that, for each $a \in X$, $a \in \Omega_a(r)$ for all $r > 0$, and $\Omega_a(r') \subset \Omega_a(r)$ if $r' \leq r$. It is clear that, for each $x \in X$, $S_\alpha(x, r') \subset S_\alpha(x, r)$ if $r' \leq r$, and $\lim_{r \rightarrow \infty} S_\alpha(x, r) = X$.

Theorem 1.1. *Let $1 \leq p < q < \infty$ and $0 < \alpha < 2\kappa$. Assume that σ and ω are Borel measures on X , which satisfy the condition: There exists a positive constant C_1 such that*

$$(1.4) \quad \left| S_\alpha \left(x, \frac{4\kappa}{\alpha}(2r+t) \right) \right|_\omega^{1/q} \left(\int_{X \setminus B(x, r)} K_t(x, y)^{p'} d\sigma(y) \right)^{1/p'} \leq C_1,$$

if $1 < p < \infty$, or

$$(1.5) \quad \left| S_\alpha \left(x, \frac{4\kappa}{\alpha}(2r+t) \right) \right|_\omega^{1/q} \text{ess sup}_\sigma \{K_t(x, y): y \in X \setminus B(x, r)\} \leq C_1,$$

if $p = 1$, for all $(x, t) \in \hat{X}$ and $r \geq 0$, where ess sup_σ denotes the essential supremum with respect to the measure σ .

Then the weak-type inequality

$$(1.6) \quad \left| \left\{ a \in X: \sup_{(x, t) \in \Omega_a} T(f d\sigma)(x, t) > \lambda \right\} \right|_\omega^{1/q} \leq \frac{C}{\lambda} \left(\int_X f(x)^p d\sigma(x) \right)^{1/p}$$

holds for all σ -measurable functions $f \geq 0$.

We shall say that an operator T defined by (1.2) is an operator of potential type if its kernel K satisfies the condition: For a fixed constant $0 < \alpha < 2\kappa$, there exists a positive constant C_0 such that

(1.7)

$$K_t(x, y) \leq C_0 K_{t'}(x', y) \text{ whenever } (x', t') \in \widehat{B}\left(x, \frac{8\kappa^2}{\alpha}(r+t)\right) \text{ and } y \in X \setminus B(x, r),$$

for all $(x, t) \in \widehat{X}$ and $r \geq 0$. (See also [8] for an essentially equivalent definition.)

Theorem 1.2. *Let $1 \leq p < q < \infty$ and $0 < \alpha < 2\kappa$, and let σ and ω be locally finite Borel measures on X . Assume that the kernel K satisfies condition (1.7); then the weak-type inequality (1.6) holds for all σ -measurable functions $f \geq 0$ if and only if there exists a positive constant C_1 such that*

$$(1.8) \quad \left| S_\alpha\left(x, \frac{4\kappa}{\alpha}t\right) \right|_\omega^{1/q} \left(\int_X K_t(x, y)^{p'} d\sigma(y) \right)^{1/p'} \leq C_1,$$

if $1 < p < \infty$, or

$$(1.9) \quad \left| S_\alpha\left(x, \frac{4\kappa}{\alpha}t\right) \right|_\omega^{1/q} \text{ess sup}_\sigma \{K_t(x, y) : y \in X\} \leq C_1,$$

if $p = 1$, for all $(x, t) \in \widehat{X}$.

2. PROOF OF THEOREM 1.1

We shall assume that $1 < p < \infty$. For $p = 1$, only some mild modifications of the following proof are needed.

For any fixed function $f \geq 0$ and number $\lambda > 0$, we set

$$(2.1) \quad E_\lambda = \left\{ a \in X : \sup_{(x,t) \in \Omega_a} T(f d\sigma)(x, t) > \lambda \right\}.$$

Let B_R in X be a ball centered at some fixed $x_0 \in X$ with radius $R > 0$. We consider the sets

$$(2.2) \quad E_\lambda^*(R) = \left\{ x \in B_R : \sup_{0 < t \leq R} T(f d\sigma)(x, t) > \lambda \right\},$$

and

$$(2.3) \quad E_\lambda(R) = \left\{ a \in X : T(f d\sigma)(x, t) > \lambda \text{ for some } (x, t) \in \Omega_a, \right. \\ \left. \text{with } x \in B_R \text{ and } 0 < t \leq d(x) \right\},$$

where

$$(2.4) \quad d(x) = \sup\{t \in (0, R] : T(f d\sigma)(x, t) > \lambda\}.$$

Obviously, we have $E_\lambda^*(R) \subset E_\lambda(R) \subset E_\lambda$, and $\lim_{R \rightarrow \infty} E_\lambda(R) = E_\lambda$.

We first claim that, for each $x \in E_\lambda^*(R)$ there exists $r(x) \geq (\alpha/2\kappa)d(x)$ (we recall that $\alpha < 2\kappa$) such that

$$(2.5) \quad \left| S_\alpha\left(x, \frac{4\kappa}{\alpha}r(x)\right) \right|_\omega \leq \frac{C}{\lambda^q} \|f\|_{L^p(d\sigma)}^{q-p} \int_{B(x, r(x))} f(y)^p d\sigma(y),$$

where the constant C is independent of x and R .

Assuming the last claim for a moment, we finish the proof of the theorem. We may assume that $\sup \{r(x) : x \in E_\lambda^*(R)\} < \infty$; otherwise (1.6) would follow trivially from

$$|X|_\omega = \sup_{x \in E_\lambda^*(R)} \left| S_\alpha \left(x, \frac{4\kappa}{\alpha} r(x) \right) \right|_\omega \leq \frac{C}{\lambda^q} \|f\|_{L^p(d\sigma)}^{q-p} \int_X f(x)^p d\sigma(x).$$

Applying the covering lemma for homogeneous spaces (see [2]) with the family of balls $\{B(x, r(x)) : x \in E_\lambda^*(R)\}$, we obtain a countable, pairwise disjoint subfamily $\{B(x_k, r_k)\}_{k=0}^\infty$, where $r_k = r(x_k)$, so that $E_\lambda^*(R) \subset \bigcup_k B(x_k, 4\kappa r_k)$, and $r(x) \leq 2r_k$ for all $x \in B(x_k, 4\kappa r_k)$.

We next claim that

$$(2.6) \quad E_\lambda(R) \subset \bigcup_k S_\alpha \left(x_k, \frac{4\kappa}{\alpha} r_k \right).$$

Indeed, if $a \in E_\lambda(R)$ then there is $(x, t) \in \Omega_a$ so that $x \in E_\lambda^*(R)$ and $t \leq d(x)$. Thus, $x \in B(x_k, 4\kappa r_k)$ and $t \leq d(x) \leq (2\kappa/\alpha)r(x) \leq (4\kappa/\alpha)r_k$ for some k . Therefore, $(x, (4\kappa/\alpha)r_k) \in \Omega_a$ by our assumption on Ω . This implies that

$$x \in \Omega_a \left(\frac{4\kappa}{\alpha} r_k \right) \cap B \left(x_k, \alpha \frac{4\kappa}{\alpha} r_k \right),$$

and therefore, $a \in S_\alpha(x_k, (4\kappa/\alpha)r_k)$.

By virtue of (2.6) and (2.5), we have

$$\begin{aligned} |E_\lambda(R)|_\omega &\leq \sum_k \left| S_\alpha \left(x_k, \frac{4\kappa}{\alpha} r_k \right) \right|_\omega \\ &\leq \frac{C}{\lambda^q} \|f\|_{L^p(d\sigma)}^{q-p} \sum_k \int_{B(x_k, r_k)} f(x)^p d\sigma(x) \\ &\leq \frac{C}{\lambda^q} \|f\|_{L^p(d\sigma)}^{q-p} \int_X f(x)^p d\sigma(x) \\ &\leq \frac{C}{\lambda^q} \left(\int_X f(x)^p d\sigma(x) \right)^{q/p}, \end{aligned}$$

where we have used the fact that $\{B(x_k, r_k)\}$ is pairwise disjoint. By letting $R \rightarrow \infty$, we obtain (1.6).

We now prove (2.5). If $x \in E_\lambda^*(R)$, then there exists t with $(\alpha/2\kappa)d(x) \leq t \leq d(x)$ so that $T(f d\sigma)(x, t) > \lambda$. To simplify the notation, we shall denote $S(r) = S_\alpha(x, (4\kappa/\alpha)r)$. Without loss of generality, we can assume that there is $r > 0$ so that

$$(2.7) \quad \frac{C_1}{|S(2r+t)|_\omega^{1/q}} \left(\int_X \left(\frac{2f(y)}{\lambda} \right)^p d\sigma(y) \right)^{1/p} < 1.$$

Otherwise, (1.6) would follow trivially from

$$|X|_\omega^{1/q} = \lim_{r \rightarrow \infty} |S(2r+t)|_\omega^{1/q} \leq \frac{2C_1}{\lambda} \left(\int_X f(y)^p d\sigma(y) \right)^{1/p}.$$

Let d_0 be the greatest lower bound for the r satisfying (2.7); then, by using Hölder's inequality and condition (1.4), we have

$$\begin{aligned} \int_{X \setminus B(x,r)} K_t(x,y) f(y) d\sigma(y) &= \frac{\lambda}{2} \int_{X \setminus B(x,r)} K_t(x,y) \frac{2f(y)}{\lambda} d\sigma(y) \\ &\leq \frac{\lambda}{2} \left(\int_{X \setminus B(x,r)} K_t(x,y)^{p'} d\sigma(y) \right)^{1/p'} \left(\int_{X \setminus B(x,r)} \left(\frac{2f(y)}{\lambda} \right)^p d\sigma(y) \right)^{1/p} \\ &\leq \frac{C_1 \lambda}{2|S(2r+t)|_\omega^{1/q}} \left(\int_X \left(\frac{2f(y)}{\lambda} \right)^p d\sigma(y) \right)^{1/p} \leq \frac{\lambda}{2}, \end{aligned}$$

for all $r > d_0$. Therefore, we obtain either

$$(2.8) \quad \int_{B(x,r)} K_t(x,y) f(y) d\sigma(y) > \frac{\lambda}{2}, \quad \text{if } r > d_0 > 0,$$

or

$$(2.9) \quad K_t(x,x) f(x) |\{x\}|_\sigma > \frac{\lambda}{2}, \quad \text{if } d_0 = 0.$$

If $d_0 = 0$ then

$$\begin{aligned} \frac{\lambda}{2} &< K_t(x,x) f(x) |\{x\}|_\sigma \leq \int_{B(x,t)} K_t(x,y) f(y) d\sigma(y) \\ &\leq \left(\int_X K_t(x,y)^{p'} d\sigma(y) \right)^{1/p'} \left(\int_{B(x,t)} f(y)^p d\sigma(y) \right)^{1/p} \\ &\leq \frac{C_1}{|S(t)|_\omega^{1/q}} \left(\int_{B(x,t)} f(y)^p d\sigma(y) \right)^{1/p}, \end{aligned}$$

and so that

$$\begin{aligned} |S(t)|_\omega &\leq \left(\frac{2C_1}{\lambda} \right)^q \left(\int_{B(x,t)} f(y)^p d\sigma(y) \right)^{q/p} \\ &\leq \frac{C}{\lambda^q} \|f\|_{L^p(d\sigma)}^{q-p} \int_{B(x,t)} f(y)^p d\sigma(y). \end{aligned}$$

If $d_0 > 0$ then there is $r_0 > 0$ such that

$$(2.10) \quad \frac{C_1}{|S(2r_0+t)|_\omega^{1/q}} \left(\int_X \left(\frac{2f(y)}{\lambda} \right)^p d\sigma(y) \right)^{1/p} < 1,$$

and

$$(2.11) \quad \frac{C_1}{|S(r_0+t)|_\omega^{1/q}} \left(\int_X \left(\frac{2f(y)}{\lambda} \right)^p d\sigma(y) \right)^{1/p} \geq 1.$$

We now choose a (finite or infinite) decreasing sequence $\{r_j\}_{j=1}^n$ of positive numbers such that

$$(2.12) \quad |S(r_j+t)|_\omega \leq 2^{-j} |S(r_0+t)|_\omega \leq |S(2r_j+t)|_\omega$$

for $j = 1, 2, \dots, n$. Due to $p < q$, we have

$$\sum_{j=1}^n \left(\frac{|S(r_j + t)|_\omega}{|S(r_0 + t)|_\omega} \right)^{1/p-1/q} \leq \sum_{j=1}^{\infty} \left(\frac{1}{2^j} \right)^{1/p-1/q} = \beta < \infty.$$

Thus, we obtain from (2.8) that

$$\frac{\lambda}{2\beta} \sum_{j=1}^n \left(\frac{|S(r_j + t)|_\omega}{|S(r_0 + t)|_\omega} \right)^{1/p-1/q} \leq \sum_{j=1}^n \int_{B_j \setminus B_{j+1}} K_t(x, y) f(y) d\sigma(y),$$

where $B_j = B(x, r_j)$, and $B_{n+1} = \{x\}$ if n is finite. Hence, there exists j_0 such that

$$\begin{aligned} & \frac{\lambda}{2\beta} \left(\frac{|S(r_{j_0} + t)|_\omega}{|S(r_0 + t)|_\omega} \right)^{1/p-1/q} < \int_{B_{j_0} \setminus B_{j_0+1}} K_t(x, y) f(y) d\sigma(y) \\ & \leq \left(\int_{B_{j_0} \setminus B_{j_0+1}} K_t(x, y)^{p'} d\sigma(y) \right)^{1/p'} \left(\int_{B_{j_0} \setminus B_{j_0+1}} f(y)^p d\sigma(y) \right)^{1/p} \\ & \leq \frac{C_1}{|S(2r_{j_0+1} + t)|_\omega^{1/q}} \left(\int_{B_{j_0}} f(y)^p d\sigma(y) \right)^{1/p} \\ & \leq \frac{2^{1/q} C_1}{|S(r_{j_0} + t)|_\omega^{1/q}} \left(\int_{B_{j_0}} f(y)^p d\sigma(y) \right)^{1/p}, \end{aligned}$$

the last step follows from (2.12). Consequently, by using (2.11), we get

$$\begin{aligned} |S(r_{j_0} + t)|_\omega & \leq \frac{C}{\lambda^p} |S(r_0 + t)|_\omega^{1-p/q} \int_{B(x, r_{j_0})} f(y)^p d\sigma(y) \\ & \leq \frac{C}{\lambda^q} \|f\|_{L^p(d\sigma)}^{q-p} \int_{B(x, r_{j_0})} f(y)^p d\sigma(y) \\ & \leq \frac{C}{\lambda^q} \|f\|_{L^p(d\sigma)}^{q-p} \int_{B(x, r_{j_0}+t)} f(y)^p d\sigma(y). \end{aligned}$$

This completes the proof of (2.5), if we choose $r(x) = r_{j_0} + t$ and note $t \geq (\alpha/2\kappa)d(x)$. Therefore Theorem 1.1 is proved.

3. PROOF OF THEOREM 1.2

We first show that conditions (1.8) and (1.9) imply conditions (1.4) and (1.5), respectively, if the kernel K satisfies condition (1.7). Then the sufficient part of the theorem follows from Theorem 1.1. To see this, we fix $x \in X$, $t > 0$ and $r \geq 0$. Then, $(x, 2r + t) \in \widehat{B}(x, (8\kappa^2/\alpha)(r + t))$, since $0 < \alpha < 2\kappa$, and so that

$$K_t(x, y) \leq C_0 K_{2r+t}(x, y)$$

for $y \in X \setminus B(x, r)$, by (1.7). Therefore

$$\begin{aligned} & \left| S_\alpha \left(x, \frac{4\kappa}{\alpha}(2r+t) \right) \right|_\omega^{1/q} \left(\int_{X \setminus B(x, r)} K_t(x, y)^{p'} d\sigma(y) \right)^{1/p'} \\ & \leq \left| S_\alpha \left(x, \frac{4\kappa}{\alpha}(2r+t) \right) \right|_\omega^{1/q} \left(\int_{X \setminus B(x, r)} K_{2r+t}(x, y)^{p'} d\sigma(y) \right)^{1/p'} \\ & \leq \left| S_\alpha \left(x, \frac{4\kappa}{\alpha}t \right) \right|_\omega^{1/q} \left(\int_X K_t(x, y)^{p'} d\sigma(y) \right)^{1/p'} \leq C_1. \end{aligned}$$

The proof for the case $p = 1$ is similar.

We now prove the necessity part of the theorem. We consider the case of $1 < p < \infty$ first. Let $(x_0, t_0) \in \widehat{X}$ be temporarily fixed, and for each positive integer N , define

$$K_{t_0}^N(x_0, y) = \begin{cases} K_{t_0}(x_0, y) & \text{if } K(x_0, y) < N, \\ N & \text{if } K_{t_0}(x_0, y) \geq N. \end{cases}$$

Let $R > 0$ be arbitrary fixed constant and $0 < \beta < 1/C_0$, we now take

$$f(y) = K_{t_0}^N(x_0, y)^{p'/p} \chi_{B(x_0, R)}(y)$$

and $\lambda = \beta \int_{B(x_0, R)} K_{t_0}^N(x_0, y)^{p'} d\sigma(y)$. We note that $\lambda < \infty$ since σ is locally finite. If $a \in S_\alpha(x_0, (4\kappa/\alpha)t_0)$, then there is $x \in \Omega_a((4\kappa/\alpha)t_0) \cap B(x_0, 4\kappa t_0)$, and hence $(x, (4\kappa/\alpha)t_0) \in \Omega_a \cap \widehat{B}(x_0, (8\kappa^2/\alpha)t_0)$, since $0 < \alpha < 2\kappa$. Apply condition (1.7) with $r = 0$, we get $K_{t_0}(x_0, y) \leq C_0 K_{(4\kappa/\alpha)t_0}(x, y)$ for all $y \in X$, and therefore, $K_{t_0}^N(x_0, y) \leq C_0 K_{(4\kappa/\alpha)t_0}(x, y)$ for all $y \in X$. Thus

$$T(f d\sigma) \left(x, \frac{4\kappa}{\alpha}t_0 \right) \geq \frac{1}{C_0} \int_{B(x_0, R)} K_{t_0}^N(x_0, y) K_{t_0}^N(x_0, y)^{p'/p} d\sigma(y) > \lambda,$$

which implies that

$$S_\alpha \left(x_0, \frac{4\kappa}{\alpha}t_0 \right) \subset \left\{ a \in X : \sup_{(x, t) \in \Omega_a} T(f d\sigma)(x, t) > \lambda \right\}.$$

Therefore, by using the weak-type inequality (1.6), we obtain

$$\begin{aligned} & \beta \left| S_\alpha \left(x_0, \frac{4\kappa}{\alpha}t_0 \right) \right|_\omega^{1/q} \int_{B(x_0, R)} K_{t_0}^N(x_0, y)^{p'} d\sigma(y) \\ & \leq \lambda \left| \left\{ a \in X : \sup_{(x, t) \in \Omega_a} T(f d\sigma)(x, t) > \lambda \right\} \right|_\omega^{1/q} \\ & \leq C \left(\int_{B(x_0, R)} K_{t_0}^N(x_0, y)^{p'} d\sigma(y) \right)^{1/p}. \end{aligned}$$

Thus, (1.8) follows by division, and then letting $R \rightarrow \infty$ and $N \rightarrow \infty$.

For the case $p = 1$, we fix $(x_0, t_0) \in \widehat{X}$ temporarily, and for each positive integer $N > 4\kappa t_0$ set

$$\Lambda_N = \min \{ N, \text{ess sup}_\sigma \{ K_{t_0}(x_0, y) : y \in X \} \}.$$

We could assume that $\Lambda_N > 0$, otherwise (1.9) would hold trivially. Let R be an arbitrary positive number, and choose a number $0 < \eta < 1$ so that the set

$$U_{R,N} = \{y \in B(x_0, N) : K_{t_0}(x_0, y) \geq \eta \Lambda_N\}$$

has nonzero and finite σ -measure, since σ is locally finite. Now, take $f = \chi_{U_{R,N}}$ and $\lambda = \beta \Lambda_N |U_{R,N}|_\sigma$ with $0 < \beta < \eta/C_0$. As before, if $a \in S_\alpha(x_0, (4\kappa/\alpha)t_0)$, then there is $x \in \Omega_a((4\kappa/\alpha)t_0) \cap B(x_0, 4\kappa t_0)$ and hence condition (1.7) implies that

$$T(f d\sigma) \left(x, \frac{4\kappa}{\alpha} t_0 \right) \geq \frac{1}{C_0} \int_X K_{t_0}(x_0, y) \chi_{U_{R,N}}(y) d\sigma(y) \geq \frac{\eta}{C_0} \Lambda_N |U_{R,N}|_\sigma > \lambda,$$

since $N > 4\kappa t_0$. Therefore, by the weak-type inequality (1.6), we obtain

$$\begin{aligned} & \beta \left| S_\alpha \left(x_0, \frac{4\kappa}{\alpha} t_0 \right) \right|_\omega^{1/q} \Lambda |U_{R,N}|_\sigma \\ & \leq \lambda \left| \left\{ a \in X : \sup_{(x,t) \in \Omega_a} T(f d\sigma)(x, t) > \lambda \right\} \right|_\omega^{1/q} \\ & \leq C \left(\int_X \chi_{U_{R,N}}(y) d\sigma(y) \right) = C |U_{R,N}|_\sigma. \end{aligned}$$

Again, (1.9) follows by division, and then letting $R \rightarrow \infty$ and $N \rightarrow \infty$. This completes the proof of Theorem 1.2.

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