ON WEIGHTED INEQUALITIES FOR SINGULAR INTEGRALS

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ABSTRACT. In this note we consider singular integrals associated to Calderón-Zygmund kernels. We prove that if the kernel is supported in $(0, \infty)$ then the one-sided A_p condition, A_p^- , is a sufficient condition for the singular integral to be bounded in $L^p(w)$, $1 < p < \infty$, or from $L^1(wdx)$ into weak- $L^1(wdx)$ if $p = 1$. This one-sided A_p condition becomes also necessary when we require the uniform boundedness of the singular integrals associated to the dilations of a kernel which is not identically zero in $(0, \infty)$. The two-sided version of this result is also obtained: Muckenhoupt's A_p condition is necessary for the uniform boundedness of the singular integrals associated to the dilations of a general Calderón-Zygmund kernel which is not the function zero either in $(-\infty, 0)$ or in $(0, \infty)$.

INTRODUCTION

It is a classical result in the theory of weighted inequalities the fact that the A_p condition of B. Muckenhoupt on a weight w is equivalent to the $L^p(wdx)$ boundedness of the Hilbert transform. This result was proved in 1973 by Hunt, Muckenhoupt and Wheeden [HMW]. In 1974 Coifman and Fefferman [CF] gave a different proof which relies on a good- λ inequality, producing an integral estimate of the singular integral in terms of the Hardy-Littlewood maximal operator.

Since 1986 the work by E. Sawyer [S], Andersen and Sawyer [AS], Martín Reyes, Ortega Salvador and de la Torre [MOT], [MT] has shown that many positive operators of real analysis have one-sided versions for which the classes of weights are larger than Muckenhoupt's ones. Our purpose here is to study the corresponding problems for singular integrals.

The situation for one-sided singular integrals is different. The symmetry properties of the Hilbert kernel produce the necessary cancellation properties of a singular integral, so that, no one-sided truncation of $1/x$ is expected to produce a one-sided singular integral. Nevertheless, as we show in Lemma (1.5), the class of general singular integral Calderón-Zygmund kernels supported on a half line is nontrivial. We ask for the more general class of weights w for which such singular integral operators are bounded in $L^p(wdx)$. It turns out (Theorem (2.1)) that the one-sided A_p condition is a sufficient condition which becomes also necessary when we require

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the uniform boundedness of the singular integrals associated to the dilations of a kernel which is not the function zero in $(0, \infty)$ or in $(-\infty, 0)$ (Theorem (2.6)). The two-sided version of this result gives (see also Theorem (2.6)) that Muckenhoupt's A_p condition is necessary for the uniform boundedness of the singular integrals associated to the dilations of a Calderón-Zygmund kernel which is not the function zero either in $(-\infty, 0)$ or in $(0, \infty)$.

1. ONE-SIDED SINGULAR INTEGRALS

We shall say that a function k in $L^1_{loc}(\mathbb{R} - \{0\})$ is a Calderón-Zygmund kernel if the following properties are satisfied:

(1.1) there exists a finite constant B_1 such that

$$\left| \int_{\varepsilon < |x| < N} k(x) dx \right| \leq B_1 \quad \text{for all } \varepsilon \text{ and all } N, \text{ with } 0 < \varepsilon < N,$$

and furthermore $\lim_{\varepsilon \rightarrow 0^+} \int_{\varepsilon < |x| < 1} k(x) dx$ exists,

(1.2) there exists a finite constant B_2 such that

$$|k(x)| \leq \frac{B_2}{|x|}, \quad \text{for all } x \neq 0,$$

(1.3) there exists a finite constant B_3 such that

$$|k(x-y) - k(x)| \leq B_3 |y| |x|^{-2} \quad \text{for all } x \text{ and } y \text{ with } |x| > 2|y| > 0.$$

These conditions are known to be sufficient for the L^p boundedness and the weak type $(1, 1)$ of the maximal operator

$$T^* f(x) = \sup_{\varepsilon > 0} |T_\varepsilon f(x)|,$$

with

$$T_\varepsilon f(x) = \int_{|x-y| > \varepsilon} k(x-y) f(y) dy$$

(see e.g. [T]). In other words

$$|\{x \in \mathbb{R} : T^* f(x) > \lambda\}| \leq \frac{C_1}{\lambda} \int_{\mathbb{R}} |f|$$

and

$$\|T^* f\|_p \leq C_p \|f\|_p$$

for $1 < p < \infty$, where $|E|$ stands for the Lebesgue measure of the set E and $\|f\|_p = (\int |f(x)|^p dx)^{1/p}$. The same inequalities hold for the singular integral

$$Tf(x) = \lim_{\varepsilon \rightarrow 0^+} T_\varepsilon f(x).$$

The next lemma shows that there exist nontrivial Calderón-Zygmund kernels with support contained in $(0, \infty)$.

(1.4) Lemma. *The function*

$$(1.5) \quad k(x) = \frac{1}{x} \cdot \frac{\sin(\log x)}{\log x} \cdot \chi_{(0, \infty)}(x),$$

is a Calderón-Zygmund kernel.

Proof. To prove (1.1), take $0 < \varepsilon < N < \infty$ and change variables in the integral $\int_{\varepsilon < |x| < N} k(x) dx$ to get

$$\int_{\log \varepsilon}^{\log N} \frac{\sin t}{t} dt,$$

which is uniformly bounded in ε and N and, for fixed N , converges to a finite limit as ε tends to zero. On the other hand, since the function $g(t) = t^{-1} \sin t$ is bounded by one, it is clear that k satisfies (1.2) with $B_2 = 1$. Let us prove (1.3). Notice that the function $g(t)$ has a bounded derivative so that

$$\frac{d}{dx} k(x) = \frac{d}{dx} \frac{1}{x} g(\log x) = \frac{1}{x^2} \{g'(\log x) - g(\log x)\},$$

from which (1.3) follows with a constant B_3 . $\square$

2. WEIGHTED INEQUALITIES

The one-sided Hardy-Littlewood maximal functions are defined by

$$M^+ f(x) = \sup_{h>0} \frac{1}{h} \int_x^{x+h} |f| \quad \text{and} \quad M^- f(x) = \sup_{h>0} \frac{1}{h} \int_{x-h}^x |f|.$$

It is a known result by E. Sawyer [S] that M^- applies $L^p(wdx)$ into $L^p(wdx)$, $1 < p < \infty$, if and only if the weight w satisfies

$$A_p^- : \left(\int_a^b w^{-\frac{1}{p-1}} \right)^{p-1} \left(\int_b^c w \right) \leq C(c-a)^p$$

for all numbers $a < b < c$ with a finite constant C independent of a, b , and c .

Also M^- applies $L^1(wdx)$ into weak- $L^1(wdx)$ if and only if w satisfies

$$A_1^- : M^+ w \leq Cw \quad \text{a.e.}$$

Analogous results hold for the operator M^+ and the corresponding A_p^+ classes which are defined in the obvious way.

In the more recent paper [MPT] the one-sided analog of A_∞ weights are studied. A weight w is in A_∞^- if and only if there exist positive numbers C and δ such that for all numbers $a < b < c$ and all measurable sets $E \subset (a, b)$

$$\frac{|E|}{c-a} \leq C \left(\frac{w(E)}{w(b,c)} \right)^\delta$$

(as usual, $w(E)$ stands for $\int_E w(x) dx$). This class of weights is proved to coincide with the union of the A_p^- classes with $p \geq 1$. In fact, we have the following result:

Theorem A [MPT, Theorem 1]. *Let w be a weight, i.e., a positive, locally integrable function in the real line. The following are equivalent:*

- (a) *There exists p , $1 \leq p < \infty$, such that $w \in A_p^-$.*
- (b) *w is in A_∞^- .*
- (c) *There exist positive numbers C and δ such that*

$$\frac{w(E)}{w(a,c)} \leq C \left(\frac{|E|}{b-a} \right)^\delta$$

for all numbers $a < b < c$ and all measurable sets $E \subset (b, c)$.

We are now in position to state the main results of this note.

(2.1) Theorem. *Let k be a Calderón-Zygmund kernel with support in $\mathbb{R}^+ = (0, \infty)$. Then*

(2.2) *given a weight w in A_∞^- there exists a constant C_p depending only on B_1 , B_2 , B_3 , p and the constants in the A_∞^- condition, such that*

$$\int_{\mathbb{R}} |T^*f(x)|^p w(x) dx \leq C_p \int_{\mathbb{R}} |M^-f(x)|^p w(x) dx, \quad 1 < p < \infty,$$

and

$$\sup_{\lambda > 0} \lambda^p w(\{x : T^*f(x) > \lambda\}) \leq C_p \sup_{\lambda > 0} \lambda^p w(\{x : M^-f(x) > \lambda\}), \quad 1 \leq p < \infty,$$

for all $f \in L^p(wdx)$,

(2.3) *given a weight $w \in A_p^-$ with $1 < p < \infty$ there exists a constant C depending only on B_1 , B_2 , B_3 , p and the constant in the condition A_p^- , such that*

$$\int_{\mathbb{R}} |T^*f(x)|^p w(x) dx \leq C \int_{\mathbb{R}} |f(x)|^p w(x) dx,$$

for all $f \in L^p(wdx)$,

(2.4) *given a weight $w \in A_1^-$ there exists a constant C depending only on B_1 , B_2 , B_3 and the constant in the condition A_1^- such that*

$$w(\{x : T^*f(x) > \lambda\}) \leq \frac{C}{\lambda} \int_{\mathbb{R}} |f(x)| w(x) dx$$

for all $f \in L^1(wdx)$ and all $\lambda > 0$.

(2.5) *Remarks.* Let us remark here that similar right-hand-sided results hold with kernels supported on the negative real numbers, changing M^- to M^+ and A_p^- to A_p^+ .

Let k be a Calderón-Zygmund kernel and let us consider for each $\alpha > 0$ the dilation

$$k_\alpha(x) = \alpha k(\alpha x).$$

It is clear that k_α is also a Calderón-Zygmund kernel with the same constants B_1 , B_2 and B_3 as k . Therefore, Theorem (2.1) gives that the maximal singular integrals T_α^* associated to the dilations k_α are uniformly bounded in $L^p(wdx)$ if $w \in A_p^-$ with $1 < p < \infty$ and from $L^1(wdx)$ into weak- $L^1(wdx)$ if $w \in A_1^-$. The next theorem is a kind of converse of this remark.

(2.6) Theorem. *Let k be a Calderón-Zygmund kernel. For each $\alpha > 0$, let T_α^* denote the maximal operator with kernel $k_\alpha(x) = \alpha k(\alpha x)$ and let $1 \leq p < \infty$. Let w be a weight and assume that the operators T_α^* are uniformly bounded from $L^p(wdx)$ into weak- $L^p(wdx)$.*

- (a) *If there exists $z > 0$ such that $k(z) \neq 0$ then $w \in A_p^-$.*
- (b) *If there exists $s < 0$ such that $k(s) \neq 0$ then $w \in A_p^+$.*
- (c) *If there exist $s < 0 < z$ such that $k(s) \neq 0 \neq k(z)$ then $w \in A_p$; i.e., w belongs to the A_p class of Muckenhoupt.*

Theorems (2.1) and (2.6) hold also for the singular integral

$$Tf(x) = \lim_{\varepsilon \rightarrow 0^+} T_\varepsilon f(x).$$

The proofs for T are similar to the corresponding for T^* or follow easily from the theorem for T^* .

Proof of Theorem (2.1). Theorem (2.1) is an easy consequence of Sawyer's results for M^- [S] and of the next lemma which is itself an extension, to the one-sided setting, of the good- λ inequality of Coifman and Fefferman [CF]. $\square$

(2.7) Lemma. *Let k be a Calderón-Zygmund kernel with support in $\mathbb{R}^+$. Let w be a weight in A_∞^- . Then there exist positive constants C and γ_o such that for every $0 < \gamma \leq \gamma_o$ the inequality*

$$(2.8) \quad w(\{x \in \mathbb{R} : T^*f(x) > 2\lambda, M^-f(x) < \gamma\lambda\}) \leq C\gamma^\delta w(\{x \in \mathbb{R} : T^*f(x) > \lambda\}),$$

$f \in L^1$ and every positive λ with δ the exponent in statement (c) of Theorem A.

Proof of (2.7). Since the set $\{x : T^*f(x) > \lambda\}$ is open and has finite measure for f in L^1 , we have that it can be written as a disjoint countable union of open intervals. Let $J = (a, b)$ be such an interval. It is enough to prove that there exist C and γ_o such that

$$(2.9) \quad w(\{x \in J : T^*f(x) > 2\lambda, M^-f(x) < \gamma\lambda\}) \leq C\gamma^\delta w(J),$$

for every $0 < \gamma \leq \gamma_o$ and every $\lambda > 0$ (throughout the proof, C will be a constant that may change from line to line.) Let us now take a sequence $\{x_i : i \in \mathbb{N}\}$ in $J = (a, b)$ in such a way that $x_0 = b$ and $x_{i-1} - x_i = x_i - a$ for every $i \geq 1$. Since the weight $w \in A_\infty^-$ we have by Theorem A (statement (c) that, in order to prove (2.9), we only need to show that

$$(2.10) \quad |E_i| \leq C\gamma(x_{i+1} - x_{i+2}),$$

where $E_i = \{x \in (x_{i+1}, x_i) : T^*f(x) > 2\lambda, M^-f(x) < \gamma\lambda\}$. In fact, if (2.10) holds then by (c) in Theorem A we have that

$$w(E_i) \leq C\gamma^\delta w(x_{i+2}, x_i).$$

Summing up in i we obtain (2.9).

In order to prove (2.10) we fix $i \in \mathbb{N}$ and choose $\bar{a} < a$ in such a way that $x_i - a = a - \bar{a}$. Now decompose f as $f_1 + f_2$ with $f_1 = f\chi_{(\bar{a}, \infty)}$. If $E_i = \emptyset$ there is nothing to prove. Let us write $\xi_0 = \sup\{\xi \in (x_{i+1}, x_i) : M^-f(\xi) \leq \gamma\lambda\}$ and notice that

$$(2.11) \quad \begin{aligned} E_i \subset & \{x \in (x_{i+1}, \xi_0) : T^*f_1(x) > \lambda/2\} \\ & \cup \{x \in (x_{i+1}, x_i) : T^*f_2(x) > \frac{3}{2}\lambda, M^-f(x) \leq \gamma\lambda\}. \end{aligned}$$

Since for $x \in (x_{i+1}, \xi_0)$, $T^*f_1(x) = T^*(f\chi_{(\bar{a}, \xi_0)})(x)$ because of the support property on k , applying the weak type $(1, 1)$ of T^* we have the desired bound for the measure of the first term on the right of (2.11):

$$\begin{aligned} |\{(x_{i+1}, \xi_0) : T^*f_1 > \lambda/2\}| & \leq \frac{C}{\lambda} \int_{\bar{a}}^{\xi_0} |f| \\ & \leq \frac{C}{\lambda} (\xi_0 - \bar{a}) M^-f(\xi_0) \leq C\gamma(x_{i+1} - x_{i+2}). \end{aligned}$$

We shall now prove that the second set on the right term of (2.11) is essentially empty for γ small enough. Take $x \in (x_{i+1}, x_i)$ and ε positive. Then, with $k^{(\varepsilon)} = k\chi_{(\varepsilon, \infty)}$, we have

$$\begin{aligned} |T_\varepsilon f_2(x)| &= \left| \int_{-\infty}^x k^{(\varepsilon)}(x-y)f_2(y) dy \right| = \left| \int_{-\infty}^{\bar{a}} k^{(\varepsilon)}(x-y)f(y) dy \right| \\ &\leq \left| \int_{-\infty}^{\bar{a}} [k^{(\varepsilon)}(x-y) - k^{(\varepsilon)}(a-y)]f(y) dy \right| + \left| \int_{-\infty}^{\bar{a}} k^{(\varepsilon)}(a-y)f(y) dy \right| \\ &= \text{I} + \text{II}. \end{aligned}$$

Let us first estimate II. For $\varepsilon \geq a - \bar{a}$ we have

$$\left| \int_{-\infty}^{\bar{a}} k^{(\varepsilon)}(a-y)f(y) dy \right| = \left| \int_{-\infty}^a k^{(\varepsilon)}(a-y)f(y) dy \right| \leq T^*f(a) \leq \lambda.$$

If $\varepsilon < a - \bar{a}$ then

$$\left| \int_{-\infty}^{\bar{a}} k^{(\varepsilon)}(a-y)f(y) dy \right| = \left| \int_{-\infty}^a k^{(a-\bar{a})}(a-y)f(y) dy \right| \leq T^*f(a) \leq \lambda.$$

In order to finish the proof of the lemma, we only need to show that there exists $\gamma_0 > 0$ such that for every $0 < \gamma \leq \gamma_0$ and $x \in (x_{i+1}, x_i)$ with $M^-f(x) \leq \gamma\lambda$ we have $\text{I} \leq \lambda/2$. Let $\chi_\varepsilon(t)$ denote the characteristic function of the half line (ε, ∞) . Then

$$\begin{aligned} \text{I} &\leq \int_{-\infty}^{\bar{a}} |k(x-y)\chi_\varepsilon(x-y) - k(a-y)\chi_\varepsilon(a-y)| |f(y)| dy \\ &\leq \int_{-\infty}^{\bar{a}} |k(x-y) - k(a-y)| \chi_\varepsilon(a-y) |f(y)| dy \\ &\quad + \int_{-\infty}^{\bar{a}} |k(x-y)| |\chi_\varepsilon(x-y) - \chi_\varepsilon(a-y)| |f(y)| dy \\ &= \text{III} + \text{IV}. \end{aligned}$$

Observe that $x-y > 2(x-a)$ for all $y < \bar{a}$. Then applying the regularity condition (1.3) we can estimate III in the following way:

$$\begin{aligned} \text{III} &\leq B_3 \int_{-\infty}^{\bar{a}} \frac{(x-a)}{(x-y)^2} |f(y)| dy \\ &\leq C(x_i - x_{i+1}) \int_{-\infty}^{\bar{a}} \frac{|f(y)|}{(a-y)^2} dy \\ &\leq C(x_i - x_{i+1}) \sum_{j=0}^{\infty} \int_{a-(a-\bar{a})2^{j+1}}^{a-(a-\bar{a})2^j} \frac{|f(y)|}{(a-y)^2} dy \\ &\leq C(x_i - x_{i+1}) \sum_{j=0}^{\infty} \frac{1}{((a-\bar{a})2^j)^2} \int_{a-(a-\bar{a})2^{j+1}}^{\xi_0} |f(y)| dy \\ &\leq \tilde{C}M^-f(\xi_0) \leq \tilde{C}\gamma\lambda < \lambda/4, \end{aligned}$$

if $\gamma_0 < 1/(4\tilde{C})$. On the other hand, IV can be estimated applying the size condition on k given by (1.2),

$$\begin{aligned} \text{IV} &\leq B \int_{-\infty}^{\bar{a}} \frac{1}{x-y} |\chi_\varepsilon(x-y) - \chi_\varepsilon(a-y)| |f(y)| dy \\ &= B \int_{(-\infty, \bar{a}) \cap (a-\varepsilon, x-\varepsilon)} \frac{|f(y)|}{x-y} dy. \end{aligned}$$

The last term is zero for $\varepsilon \leq a - \bar{a}$ since the domain is empty. For $\varepsilon > a - \bar{a}$ it is bounded by a constant times $M^-f(\xi_0)$. Choosing γ_0 small enough we also have $\text{IV} \leq \lambda/4$ and the lemma is proved. $\square$

Proof of Theorem (2.6). We shall only prove statement (a) since the proof of (b) is analogous and (c) follows from (a) and (b) because $A_p^+ \cap A_p^- = A_p$.

Let $1 \leq p < \infty$ fixed. Assume that there exists a constant C such that the inequality

$$w(\{x \in \mathbb{R} : |T_\alpha^* f(x)| > \lambda\}) \leq \frac{C}{\lambda^p} \int_{\mathbb{R}} |f(x)|^p w(x) dx$$

holds for every $\lambda > 0$, every $\alpha > 0$ and all measurable f .

We may assume without loss of generality that $k(z) > 0$. Since the kernel is continuous in z by (1.3), there exist positive numbers δ , z_1 and z_2 with $z_1 < z_2$ such that

$$[z_1, z_2] \subset \{x : k(x) > \delta\}.$$

Let $\gamma = z_1/z_2$ and $\beta = (1 - \gamma)/2$. Let $a < d$ and let us take b and c such that $a < b < c < d$ with $c - b = \gamma(d - a)$ and $d - c = b - a = \beta(d - a)$. Now choose $\alpha = z_2/(d - a) = z_1/(c - b)$. Then

$$z_1 = \alpha(c - b) \leq \alpha(x - y) \leq \alpha(d - a) = z_2$$

for all $x \in (c, d)$ and all $y \in (a, b)$. Therefore, if $x \in (c, d)$, $f = \sigma \chi_{(a,b)}$, $\sigma = w^{-\frac{1}{p-1}}$, we obtain

$$|T_\alpha^* f(x)| \geq \left| \int_a^b \sigma(y) k_\alpha(x - y) dy \right| \geq \alpha \delta \int_a^b \sigma(y) dy.$$

Since the maximal operators T_α^* are uniformly bounded we get

$$\int_c^d w \leq \frac{C}{(\delta\alpha)^p} \left(\int_a^b \sigma \right)^{1-p} = \frac{C(d-a)^p}{\delta^p z_2^p} \left(\int_a^b \sigma \right)^{1-p},$$

and therefore, keeping in mind that δ and z_2 are constants which depend only on the kernel k , we have

$$(2.12) \quad \int_c^d w \leq C(d-c) \left(\frac{b-a}{\int_a^b \sigma} \right)^{p-1} \leq C(d-c) (M_\sigma(\sigma^{-1} \chi_{(a,d)})(a))^{p-1},$$

where

$$M_\sigma f(x) = \sup_{h,k>0} \frac{\int_{x-h}^{x+k} |f| \sigma}{\int_{x-h}^{x+k} \sigma}.$$

Now, let $x_0 = d$ and for all nonnegative integer i , let x_{i+1} be such that $x_i - x_{i+1} = \beta(x_i - a)$. Then inequality (2.12) applied to a , x_{i+1} , x_i , gives

$$\begin{aligned} \int_{x_{i+1}}^{x_i} w &\leq C(x_i - x_{i+1}) (M_\sigma(\sigma^{-1}\chi_{(a,x_i)})(a))^{p-1} \\ &\leq C(x_i - x_{i+1}) (M_\sigma(\sigma^{-1}\chi_{(a,d)})(a))^{p-1}. \end{aligned}$$

Summing up in i and dividing by $d - a$

$$\frac{1}{d-a} \int_a^d w \leq C (M_\sigma(\sigma^{-1}\chi_{(a,d)})(a))^{p-1}.$$

Now, it follows easily from this inequality that

$$M(w\chi_{(a,d)})(a) \leq C (M_\sigma(\sigma^{-1}\chi_{(a,d)})(a))^{p-1},$$

which is equivalent to $w \in A_p^-$ (see Theorem 2 in [MPT] for A_p^- weights). $\square$

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